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ON CONTINUUM MANY TOPOLOGICALLY DIFFERENT COMPLETE CONNECTED STRICTLY CONVEX HYPERSURFACES IN THE HYPERBOLIC SPACE THAT ARE OBTAINED BY GLUING TWO COPIES OF A FIXED MANIFOLD WITH BOUNDARY

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We prove the existence of a family $\mathcal{M}(n)$ $(n \geq 2)$ of complete connected strictly convex n-dimensional surfaces in the $n+1$-dimensional hyperbolic space $\mathbf{H}^{n+1}$ (boundaries of strictly convex bodies in $\mathbf{H}^{n+1}$) such that 1) *card* $\mathcal{M}(n) = c$ (where c is the cardinality of the continuum); 2) if $M_i \in \mathcal{M}(n)$ $(i = 1, 2)$, $M_1 \neq M_2$, and, for each i, $S_i \subset M_i$ is a compact set such that *card* $S_i \leq \aleph_0$, then $M_1 \setminus S_1$ and $M_2 \setminus S_2$ are not homeomorphic; 3) there is a connected n-dimensional topological manifold $G(n)$ with boundary (in case $n \geq 3$ the boundary of $G(n)$ is connected) such that each $M \in \mathcal{M}(n)$ is a topological manifold obtained by gluing two copies of $G(n)$ along their boundaries.

To the memory of my Teacher
Alexander Danilovich Alexandrov

Introduction

In this paper we consider the topological diversity of complete connected strictly convex n-dimensional $(n \geq 2)$ surfaces in the $n+1$-dimensional hyperbolic space $\mathbf{H}^{n+1}$.

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As usual, *a convex body* $B \subset \mathbf{H}^{n+1}$ (respectively, $B \subset \mathbf{E}^{n+1}$, where $\mathbf{E}^{n+1}$ is the $n+1$-dimensional Euclidean space) is a closed convex $n+1$-dimensional set.

A convex body B is called *strictly convex* if the boundary ∂B of the body B does not contain any straight-line segment and $\partial B \neq \emptyset$.

In [3, 4] we have shown how topologically different unbounded convex (and strictly convex) bodies in $\mathbf{H}^{n+1}$ (whose boundaries are not connected) might be.

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We call a set $M \subset \mathbf{H}^{n+1}$ (respectively, $M \subset \mathbf{E}^{n+1}$) *a strictly convex n-dimensional surface* if there is a strictly convex body $B(M) \subset \mathbf{H}^{n+1}$ (respectively, $B(M) \subset \mathbf{E}^{n+1}$) such that $M \subset \partial B(M)$, the set M is open (in the topology of $\partial B(M)$), and $M \neq \emptyset$.

It follows from [1, p. 73] that every complete strictly convex n-dimensional surface in the Euclidean space $\mathbf{E}^{n+1}$ is homeomorphic either to the sphere $\mathbf{S}^n$ or to the space $\mathbf{E}^n$.

Note that all strictly convex n-dimensional surfaces of the family $\mathcal{M}(n)$ (see the statement of the Theorem) are (connected) boundaries of strictly convex bodies in $\mathbf{H}^{n+1}$; hence they are complete.

As usual, $convS$ is a convex hull of a set S; $cardS$ is the cardinality of S; $c := card\, \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers; $\aleph_0 := card\, \mathbf{N}$, where $\mathbf{N}$ is the set of positive integers.

Let us fix a hyperplane $P_0 \subset \mathbf{H}^{n+1}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ ($\mathcal{H}_1 \neq \mathcal{H}_2$) be closed half-spaces of $\mathbf{H}^{n+1}$ such that P_0 is their common boundary.

Theorem 1. *There is a family $\mathcal{M}(n)$, $n \geq 2$, of complete connected strictly convex n-dimensional surfaces in the space $\mathbf{H}^{n+1}$ such that*

1) $card\, \mathcal{M}(n) = c$;

2) for each $M \in \mathcal{M}(n)$, $convM$ is a strictly convex body and M is the boundary of $convM$;

3) if $M_i \in \mathcal{M}(n)$ $(i = 1, 2)$, $M_1 \neq M_2$, and, for each i, $S_i \subset M_i$ is a compact set such that $card\, S_i \leq \aleph_0$, then $M_1 \setminus S_1$ and $M_2 \setminus S_2$ are not homeomorphic;

4) there is a connected n-dimensional topological manifold $G(n)$ with boundary such that

a) each $M \in \mathcal{M}(n)$ is a topological manifold obtained by gluing two copies of $G(n)$ along their boundaries; moreover, for each $M \in \mathcal{M}(n)$, the intersections $M \cap \mathcal{H}_1$ and $M \cap \mathcal{H}_2$ are homeomorphic to $G(n)$;

b) in case $n \geq 3$ the boundary of $G(n)$ is connected.

Remark 1. Thus, on the one hand, the manifolds from $\mathcal{M}(n)$ are, in particular, pairwise not homeomorphic but, on the other hand, they are topologically "similar" (they are obtained by gluing two copies of $G(n)$ along their boundaries).

Proof. As usual, *a region* in the space $\mathbf{R}^n$ is a nonempty connected open set. We need the following lemma.

Lemma 1. *There is a family $\mathcal{U}(n)$, $n \geq 2$, of regions in $\mathbf{R}^n$ such that*

1) $card\, \mathcal{U}(n) = c$;

2) if $U_i \in \mathcal{U}(n)$ $(i = 1, 2)$, $U_1 \neq U_2$, and, for each i, $S_i \subset U_i$ is a compact set such that $card\, S_i \leq \aleph_0$, then $U_1 \setminus S_1$ and $U_2 \setminus S_2$ are not homeomorphic;

3) there is a connected n-dimensional topological manifold $G(n) \subset \mathbf{R}^n$ with boundary such that

a) each $U \in \mathcal{U}(n)$ is a topological manifold obtained by gluing two copies of $G(n)$ along their boundaries; moreover, for each $U \in \mathcal{U}(n)$, the intersections

$$U \cap \{(x_1, ..., x_n) : x_1 \leq 0\} \quad \textit{and} \quad U \cap \{(x_1, ..., x_n) : x_1 \geq 0\}$$

are homeomorphic to $G(n)$;

b) in case $n \geq 3$ the boundary of $G(n)$ is connected.

Proof. In [5, p. 112-113] we have defined points of rank k ($k \in \mathbf{N} \cup \{0\}$) of a topological space and, for each $k \in \mathbf{N}$, have defined a set $A(k) \subset [0,1]$.

Let $C \subset [0,1]$ be the Cantor perfect set. Put $C^- := \{-x : x \in C\}$ and $B(k) := C^- \cup A(k)$.

Let $I_k \subset [0,1]$ ($k \in \mathbf{N}$) be open intervals such that $C = [0,1] \setminus \bigcup_{k=1}^{\infty} I_k$; denote by $\mathcal{I}$ the set of all intervals I_k, $k \in \mathbf{N}$. Let $C^* \subset C$ be the set of all points of C which are not the ends of the intervals I_k, $k \in \mathbf{N}$, and are not the ends of $[0,1]$.

Let us fix a point $x_0 \in C^*$.

Like in [2, p. 101], for each point $x \in C^*$, there is a homeomorphism

$$\phi_x : [0, x_0] \to [0, x]$$

such that $\phi_x(0) = 0$, $\phi_x(x_0) = x$, and $\phi_x(C \cap [0, x_0]) = C \cap [0, x]$ (note that this statement is a consequence of the existence of a bijective mapping ϕ_x^* of the set of all intervals from $\mathcal{I}$ contained in $[0, x_0]$ onto the set of all intervals from $\mathcal{I}$ contained in $[0, x]$ such that ϕ_x^* preserves the order of the intervals).

Hence, for each point $x \in C^*$, there is a homeomorphism $\psi_x : \mathbf{R} \to \mathbf{R}$ such that $\psi_x(C) = C$, $\psi_x(x_0) = x$, for each $I \in \mathcal{I}$, the restriction of ψ_x to the interval I is a superposition of a linear mapping and a translation, and outside the open interval $]0, 1[$ the mapping ψ_x is the identity.

Let $\mu : \mathbf{R} \to \mathbf{R}^n$ be the isometry such that $\mu(0) = (0, ..., 0)$ and $\mu(1) = (0, 1, 0, ..., 0)$ (so $\mu(\mathbf{R})$ is the x_2-axis). Let $p : \mathbf{R}^n \to \mu(\mathbf{R})$ be the orthogonal projection.

For each $k \in \mathbf{N}$, let $s_k \subset \{(x_1, ..., x_n) : x_1 > 0, x_3 = 0, ..., x_n = 0\}$ be a closed straight line interval such that s_k is parallel to $\mu(I_k)$, $p(s_k) \subset \mu(I_k)$, and the distance between s_k and $\mu(I_k)$ is less than the length of I_k.

For each $k \in \mathbf{N}$, let $\tau_k : [-1, 1] \to s_k$ be a bijective superposition of a linear mapping and a translation.

Hence, for each $k \in \mathbf{N}$, the set $\tau_k(B(k))$ is compact and nowhere dense (in s_k).

Put $\Omega := \bigcup_{k=1}^{\infty} \tau_k(B(k))$. Put $G(n) := \{(x_1, ..., x_n) : x_1 \geq 0\} \setminus (\mu(C) \cup \Omega)$.

Obviously, $G(n)$ (with the topology induced from $\mathbf{R}^n$) is an n-dimensional connected topological manifold with boundary.

Note that the boundary of $G(n)$ is $\{(x_1, ..., x_n) : x_1 = 0\} \setminus \mu(C)$.

Put $C^*(x_0) := \{x : x \in C^*, x > x_0\}$.

For each $x \in C^*(x_0)$, let us define a homeomorphism $\Psi_x : \mathbf{R}^n \to \mathbf{R}^n$ as follows:

$$\Psi_x((x_1, x_2, x_3, ..., x_n)) = (x_1, \psi_x(x_2), x_3, ..., x_n).$$

Let $\eta : \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal symmetry with respect to the hyperplane $x_1 = 0$.

For each $x \in C^*(x_0)$, put $U(x) := \mathbf{R}^n \setminus (\mu(C) \cup \eta(\Omega) \cup \Psi_x(\Omega))$.

Let $\mathcal{A} \subset U(x)$ be a compact set such that *card* $\mathcal{A} \leq \aleph_0$. Obviously, the set $U(x) \setminus \mathcal{A}$ is open and connected (in particular, $U(x)$ is a region).

It is easy to see (considering, for instance, the surface of a corresponding parallelepiped) that, for any points $X_i \in \mu(C) \cup \eta(\Omega) \cup \Psi_x(\Omega) \cup \mathcal{A}$, $i = 1, 2$, $X_1 \neq X_2$, there is an $(n-1)$-dimensional topological sphere $S \subset U(x) \setminus \mathcal{A}$ (i.e. a set homeomorphic to $\mathbf{S}^{n-1}$) such that S separates X_1 from X_2 (i.e. X_1 and X_2 belong to different connected components of $(U(x) \setminus \mathcal{A}) \setminus S$).

Lemma 2. *Let $x_i \in C^*(x_0)$ $(i = 1, 2)$, $x_1 < x_2$, and, for each i, let $\mathcal{A}_i \subset U(x_i)$ be a compact set such that $\operatorname{card} \mathcal{A}_i \leq \aleph_0$. Then the sets $U(x_1) \setminus \mathcal{A}_1$ and $U(x_2) \setminus \mathcal{A}_2$ are not homeomorphic.*

Proof. Put $D^{n+1} := \{(x_1, ..., x_{n+1}) : x_1^2 + ... + x_n^2 + (x_{n+1} - 1)^2 < 1\}$ (hence D^{n+1} is an open ball in $\mathbf{R}^{n+1}$). Let S^n be the boundary of D^{n+1}. Let $O_0 \in S^n$ be the point $(0, ..., 0, 2)$.

Let $\xi : S^n \setminus \{O_0\} \to \mathbf{R}^n$ (we identify $\mathbf{R}^n$ with $\{(x_1, ..., x_{n+1}) : x_{n+1} = 0\}$) be the stereographic projection (for each $X \in S^n \setminus \{O_0\}$, the points O_0, X, and $\xi(X)$ are collinear).

For each i $(i = 1, 2)$, put $\mathcal{F}_i := \{O_0\} \cup \xi^{-1}(\mu(C) \cup \eta(\Omega) \cup \Psi_{x_i}(\Omega) \cup \mathcal{A}_i)$.

Obviously, $\mathcal{F}_i$ is a compact set. Note that O_0 is an isolated point of $\mathcal{F}_i$.

Suppose that $U(x_1) \setminus \mathcal{A}_1$ and $U(x_2) \setminus \mathcal{A}_2$ are homeomorphic. Hence there is a homeomorphism $f : S^n \setminus \mathcal{F}_1 \to S^n \setminus \mathcal{F}_2$.

We call a sequence $\{Y_k\}_{k=1}^{\infty}$ of points of the sphere S^n *i-regular* $(i = 1, 2)$ if, for each $k \in \mathbf{N}$, $Y_k \in S^n \setminus \mathcal{F}_i$, and there is a point $Y^* \in \mathcal{F}_i$ such that $Y^* = \lim_{k\to\infty} Y_k$ (in the topology of S^n).

Let us show that if a sequence $\{Y_k\}_{k=1}^{\infty}$ is 1-regular, then the sequence $\{f(Y_k)\}_{k=1}^{\infty}$ is 2-regular.

Suppose that $\{f(Y_k)\}_{k=1}^{\infty}$ is not 2-regular. Since the set $\mathcal{F}_2$ is compact and f is a homeomorphism, there are two points $Y^i \in \mathcal{F}_2$ $(i = 1, 2)$ each of which is a limit point of $\{f(Y_k)\}_{k=1}^{\infty}$.

Like it was stated earlier for $U(x)$, there is an $(n-1)$-dimensional topological sphere $S \subset S^n \setminus \mathcal{F}_2$ such that S separates Y^1 from Y^2. The set $f^{-1}(S) \subset S^n \setminus \mathcal{F}_1$ is compact. It follows from the structure of $\mathcal{F}_1$ that there is an open set $V \subset S^n \setminus f^{-1}(S)$ such that $Y^* \in V$ and the set $V \setminus \mathcal{F}_1$ is connected; thus $f(V \setminus \mathcal{F}_1)$ is connected.

There are $k_i \in \mathbf{N}$ $(i = 1, 2)$ such that, for each i, $Y_{k_i} \in V \setminus \mathcal{F}_1$ and the point $f(Y_{k_i})$ is so close to Y^i that S separates $f(Y_{k_1})$ from $f(Y_{k_2})$.

Hence $f(V \setminus \mathcal{F}_1) \cap S \neq \emptyset$, a contradiction.

Therefore the sequence $\{f(Y_k)\}_{k=1}^{\infty}$ is 2-regular.

Analogously, if a sequence $\{Y_k'\}_{k=1}^{\infty}$ is 2-regular, then $\{f^{-1}(Y_k')\}_{k=1}^{\infty}$ is 1-regular.

We define a mapping $g := \mathcal{F}_1 \to \mathcal{F}_2$ as follows: if $Y \in \mathcal{F}_1$, a sequence $\{Y_k\}_{k=1}^{\infty}$ is 1-regular, and $Y = \lim_{k\to\infty} Y_k$, then $g(Y) := \lim_{k\to\infty} f(Y_k)$.

Hence the mapping g is defined correctly and g is a bijection.

Suppose that the mapping g is not continuous. Hence there are points $Z^i \in \mathcal{F}_1$ $(i = 1, 2)$, $Z^1 \neq Z^2$, and there is a sequence $\{Z_k\}_{k=1}^{\infty}$ of points of $\mathcal{F}_1$ such that $Z^1 := \lim_{k\to\infty} Z_k$ but $\lim_{k\to\infty} g(Z_k) = g(Z^2)$. For each $k \in \mathbf{N}$, let $\{Z_m(k)\}_{m=1}^{\infty}$ be a 1-regular sequence such that $Z_k = \lim_{m\to\infty} Z_m(k)$. Hence, for each $k \in \mathbf{N}$, there is an element $Z^*(k)$ of the sequence $\{Z_m(k)\}_{m=1}^{\infty}$ such that the sequence $\{Z^*(k)\}_{k=1}^{\infty}$

is 1-regular, $Z^1 := \lim_{k\to\infty} Z^*(k)$ but $\lim_{k\to\infty} f(Z^*(k)) = g(Z^2) \neq g(Z^1)$, a contradiction.

Thus the mapping g is continuous; since $\mathcal{F}_1$ is a compact set, g is a homeomorphism.

Let $\mathbf{T} = (T, \mathcal{T})$ be a topological space (where $\mathcal{T}$ is the topology on the set T). We define a type (do not confuse with a rank) of a point $X \in T$ in the following way.

We say that a point $X \in T$ is

1) *a point of type 1 of* $\mathbf{T}$ if there is $U \in \mathcal{T}$ such that $X \in U$ and $card\ U \leq \aleph_0$;

2) *a point of type 2 of* $\mathbf{T}$ if, for each $U \in \mathcal{T}$ such that $X \in U$, there is $V \in \mathcal{T}$ such that $X \in V$, $V \subset U$, $card\ V = c$, and V does not have isolated points;

3) *a point of type 3 of* $\mathbf{T}$ if, for each $U \in \mathcal{T}$ such that $X \in U$, there is $V \in \mathcal{T}$ such that $X \in V$, $V \subset U$, and we have: each point $Y \in V \setminus \{X\}$ is either a point of type 1 of $\mathbf{T}$ or a point of type 2 of $\mathbf{T}$, and, for each i $(i = 1, 2)$, there is a point of type i of $\mathbf{T}$ in $V \setminus \{X\}$;

4) *a point of type 4 of* $\mathbf{T}$ if, for each j $(j = 1, 2, 3)$, X is not a point of type j of $\mathbf{T}$.

The mapping $g := \mathcal{F}_1 \to \mathcal{F}_2$ is a homeomorphism. Hence, for each number j $(j = 1, 2, 3, 4)$, if $X \in \mathcal{F}_1$ is a point of type j of $\mathcal{F}_1$ (we consider the set $\mathcal{F}_1$ with the topology induced from S^n and, as usual, use the same notation for the set $\mathcal{F}_1$ and for the corresponding topological space), then $g(X)$ is a point of type j of $\mathcal{F}_2$, and conversely.

For each i $(i = 1, 2)$, put

$$\mathcal{A}_i^* := \mathcal{A}_i \cup \bigcup_{k=1}^{\infty} (\eta(\tau_k(A(k) \setminus \{0\})) \cup \Psi_{x_i}(\tau_k(A(k) \setminus \{0\})))$$

and

$$\mathcal{B}_i^* := \bigcup_{k=1}^{\infty} (\eta(\tau_k(C^- \setminus \{0\})) \cup \Psi_{x_i}(\tau_k(C^- \setminus \{0\}))).$$

For each i $(i = 1, 2)$ and for each $k \in \mathbf{N}$, define a set $\mathcal{B}_i(k) \subset \mathcal{F}_i$ as follows: a point $X \in \mathcal{F}_i$ belongs to $\mathcal{B}_i(k)$ if, and only if, either $X = \eta(\tau_k(0))$ or $X = \Psi_{x_i}(\tau_k(0))$. Obviously, for each i $(i = 1, 2)$ and for each $k \in \mathbf{N}$, the set $\mathcal{B}_i(k)$ consists of two points.

For each $k \in \mathbf{N}$, the number 0 is a limit point of the set of isolated points of $A(k)$. Hence, for each i $(i = 1, 2)$, each point of $\mu(C)$ is a limit point of the set of isolated points of $\mathcal{F}_i$.

Taking this into account, it is easy to see that, for each i $(i = 1, 2)$, a point $X \in \mathcal{F}_i$ is

a) a point of type 1 of $\mathcal{F}_i$ if, and only if, $X \in \mathcal{A}_i^*$;

b) a point of type 2 of $\mathcal{F}_i$ if, and only if, $X \in \mathcal{B}_i^*$;

c) a point of type 3 of $\mathcal{F}_i$ if, and only if, $X \in \bigcup_{k=1}^{\infty} \mathcal{B}_i(k)$;

d) a point of type 4 of $\mathcal{F}_i$ if, and only if, $X \in \mu(C)$.

Note, by the way, that it is not true that, for *any* topological space $\mathbf{T}$, any point $X \in T$ is a point of at most one type of $\mathbf{T}$ (but for $\mathcal{F}_1$ and $\mathcal{F}_2$ it is true).

Since g is a homeomorphism, $g(\mathcal{A}_1^*) = \mathcal{A}_2^*$, $g(\mathcal{B}_1^*) = \mathcal{B}_2^*$, $g(\mu(C)) = \mu(C)$, and

$$g\left(\bigcup_{k=1}^{\infty} \mathcal{B}_1(k)\right) = \bigcup_{k=1}^{\infty} \mathcal{B}_2(k).$$

Since g is a homeomorphism, if $X \in \mathcal{F}_1$ is a point of rank k ($k \in \mathbf{N} \cup \{0\}$) of $\mathcal{F}_1$ (see [5, p. 112]), then $g(X)$ is a point of rank k of $\mathcal{F}_2$, and conversely.

Hence, for each $k \in \mathbf{N}$, $g(\mathcal{B}_1(k)) = \mathcal{B}_2(k)$.

By construction, there are increasing sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ of positive integers such that $\mu(x_0) = \lim_{k\to\infty} \tau_{a_k}(0)$ and $\mu(x_1) = \lim_{k\to\infty} \tau_{b_k}(0)$.

For each i ($i = 1, 2$) and for each $k \in \mathbf{N}$, $\mathcal{B}_i(a_k) = \{\eta(\tau_{a_k}(0)), \Psi_{x_i}(\tau_{a_k}(0))\}$ and $\mathcal{B}_i(b_k) = \{\eta(\tau_{b_k}(0)), \Psi_{x_i}(\tau_{b_k}(0))\}$.

For each i ($i = 1, 2$), put

$$\mathcal{K}_i := \bigcup_{k=1}^{\infty} (\mathcal{B}_i(a_k) \cup \mathcal{B}_i(b_k)).$$

Hence $g(\mathcal{K}_1) = \mathcal{K}_2$.

Obviously, the set $\mathcal{K}_1$ has three limit points ($\mu(x_0), \mu(x_1)$, and $\Psi_{x_1}(\mu(x_1))$) but the set $\mathcal{K}_2$ has four limit points ($\mu(x_0), \mu(x_1), \mu(x_2)$, and $\Psi_{x_2}(\mu(x_1))$), a contradiction.

Lemma 2 is proved. ■

Put $\mathcal{U}(n) := \{U(x) : x \in C^*(x_0)\}$. It is easy to check that the family $\mathcal{U}(n)$ enjoys properties 1) - 3) (see the statement of Lemma 1).

Lemma 1 is proved. ■

We use the model of the hyperbolic space $\mathbf{H}^{n+1}$ in the open ball D^{n+1} called Cayley-Klein (or Beltrami-Klein) model (in particular, in this model straight lines of the space $\mathbf{H}^{n+1}$ are open chords of the ball D^{n+1}).

Let $x \in C^*(x_0)$. In the proof of Lemma 1 a region $U(x)$ was defined.

Put $\mathcal{F}(x) := S^n \setminus \xi^{-1}(U(x))$. Obviously, $\mathcal{F}(x)$ is a compact set.

Let $D_0 \subset \mathbf{R}^{n+1}$ be a closed $(n+1)$-dimensional ball such that $\{O_0\} \in D_0$ and $D_0 \subset D^{n+1} \cup \{O_0\}$.

Let us define a strictly convex body $B_0(x) \subset \mathbf{R}^{n+1}$ as follows: $B_0(x)$ is the intersection of all closed balls in $\mathbf{R}^{n+1}$ each of which has radius 2 and contains the compact set $\mathcal{F}(x) \cup D_0$.

Put $B(x) := B_0(x) \cap D^{n+1}$; hence $B(x)$ is a strictly convex body in $\mathbf{H}^{n+1}$.

Let $M(x)$ be the boundary (in the topology of $\mathbf{H}^{n+1}$) of the body $B(x)$ (obviously, the topology of $\mathbf{H}^{n+1}$ coincides with the topology of the open ball D^{n+1}).

Let P_0^* be the hyperplane $x_1 = 0$ in $\mathbf{R}^n$.

Put $\mathcal{H}_1^* := \{(x_1, ..., x_n) : x_1 \leq 0\}$, $\mathcal{H}_2^* := \{(x_1, ..., x_n) : x_1 \geq 0\}$.

Without loss of generality, assume that the hyperplane $P_0 \subset \mathbf{H}^{n+1}$ coincides with $D^{n+1} \cap conv\xi^{-1}(P_0^*)$, and, for each i ($i = 1, 2$), $\mathcal{H}_i = D^{n+1} \cap conv\xi^{-1}(\mathcal{H}_i^*)$.

Put $\mathcal{M}(n) := \{M(x) : x \in C^*(x_0)\}$.

For each $x \in C^*(x_0)$, let us define a mapping $h_x : M(x) \to \xi^{-1}(U(x))$ as follows:

for each point $Y \in M(x)$, $h_x(Y) \in \xi^{-1}(U(x))$ is a point such that the points $O_0, Y, h_x(Y)$ are collinear.

Obviously, the mapping h_x is defined correctly and is a homeomorphism; the intersection $M(x) \cap \mathcal{H}_1$ is homeomorphic to $U(x) \cap \{(x_1, ..., x_n) : x_1 \leq 0\}$, and the intersection $M(x) \cap \mathcal{H}_2$ is homeomorphic to $U(x) \cap \{(x_1, ..., x_n) : x_1 \geq 0\}$.

Since the set $\mathcal{F}(x)$ is nowhere dense in S^n, for each point $X \in intB(x)$ (where $intB(x)$ is the interior of $B(x)$), there are points $X_i \in \xi^{-1}(U(x))$ $(i = 1, 2)$ such that $X \in conv\{X_1, X_2\}$. Hence there are points $X_i' \in M(x) \cap conv\{X_1, X_2\}$ $(i = 1, 2)$ such that $X \in conv\{X_1', X_2'\}$. Therefore $convM(x) = B(x)$.

Now it is easy to check that the family $\mathcal{M}(n)$ enjoys properties 1) - 4) (see the statement of the Theorem).

The Theorem is proved. ■

Remark 2. Let us show that, for each $x \in C^*(x_0)$, the strictly convex surface $M(x)$ is smooth, i.e., for each $X \in M(x)$, there is the unique hyperplane $P_X \subset \mathbf{H}^{n+1}$ of support of $M(x)$ at X.

Let $x \in C^*(x_0)$, let $X \in M(x)$.

Let $X_k \in D^{n+1} \setminus convM(x)$ $(k \in \mathbf{N})$ be such that $X = \lim_{k\to\infty} X_k$.

Denote by $\mathcal{D}$ the family of all closed balls in $\mathbf{R}^{n+1}$ each of which has radius 2 and contains the set $\mathcal{F}(x) \cup D_0$. For each $k \in \mathbf{N}$, there is $D(k) \in \mathcal{D}$ such that $X_k \notin D(k)$. Let O^* be a limit point of the set of the centers of $D(k)$, $k \in \mathbf{N}$. Let D^* be the closed ball of radius 2 with center O^*. It is easy to see that $D^* \in \mathcal{D}$ and $X \in \partial D^*$.

Suppose that there are hyperplanes $P_i \subset \mathbf{H}^{n+1}$ $(i = 1, 2)$ of support of $M(x)$ at X, $P_1 \neq P_2$. Hence there is a straight line segment $I^* \subset D^*$ such that X is an end of I^* and $(I^* \setminus \{X\}) \cap convM(x) = \emptyset$.

Let $Y_k \in I^* \setminus \{X\}$ $(k \in \mathbf{N})$ be such that $X = \lim_{k\to\infty} Y_k$. For each $k \in \mathbf{N}$, there is a ball $D'(k) \in \mathcal{D}$ such that $(I^* \setminus [X, Y_k]) \cap D'(k) = \emptyset$ (where $[X, Y_k]$ is the closed interval with ends X and Y_k). Let O^{**} be a limit point of the set of the centers of $D'(k)$, $k \in \mathbf{N}$. Let D^{**} be the closed ball of radius 2 with center O^{**}.

Obviously, $D^{**} \in \mathcal{D}$, $X \in \partial D^{**}$, and $D^{**} \neq D^*$. Hence there is a ball $D_1 \in \mathcal{D}$ such that $X \in \partial D_1$ and $D^* \cap D^{**} \setminus \{X\} \subset intD_1$. Hence $\mathcal{F}(x) \cup D_0 \subset intD_1$; therefore there is a ball $D_2 \in \mathcal{D}$ such that $X \notin D_2$, a contradiction.

Remark 3. It immediately follows from property 3 (see the statement of the Theorem) that if $M_i \in \mathcal{M}(n)$ $(i = 1, 2)$, $M_1 \neq M_2$, and, for each i, $S_i' \subset convM_i$ is a compact set such that $card\ S_i' \leq \aleph_0$, then $(convM_1) \setminus S_1'$ and $(convM_2) \setminus S_2'$ are not homeomorphic.

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