Mathematical Structures and Modeling
2012, N. 25, pp. 44–51

UDC 530.12+523.112

THE SVIDERSKIY FORMULA AND A CONTRIBUTION TO SEGAL'S CHRONOMETRY

A.V. Levichev, A.A. Akopyan

The original part of the article is a contribution to the DLF theory (for which Segal's Chronometry paved a solid mathematical foundation). The embedding of $F = U(1,1)$ into $D = U(2)$ is generalized for the $U(p,q)$ vs $U(p+q)$ case, where the Sviderskiy formula is described – as a tribute to the late Oleg S. Sviderskiy (1969–2011). The spacetime $S^1 \times SO(3)$ is introduced as underlying the Segal's compact cosmos $U(2)$, whereas the fractional linear action of $SO(3,3)$ on $SO(3)$ turns out to be a (globally defined) projective action.

1. Motivation and Introduction

A.V. Levichev. 2009

Our article is dedicated to the 100^{th} anniversary of the Academician A.D. Alexandrov (1912–1999) who, in particular, has initiated chronogeometric studies in Soviet Union. For decades he has been the leader in those studies. Having been his PhD student in 70s, the second author recalls Alexandrov's saying relevant to the current case: "There can be no better tribute to the Teacher rather than (an attempt) to discover (a piece of) new knowledge." In [3, pp.246-248], one can find more Levichev's recollections about his Teacher.

The (above) student's academic career turned out to be closely related to Irving Segal – another great name in the mathematics of the 20^{th} century. Segal's book [8] seems to be a rare exhibit in Russian libraries and it has not been translated into Russian. That is why we decided (in our Section 2) to present Segal's original views on Chronogeometry.

The original part of the article (Sections 3,4) is a contribution to the DLF theory (for which Segal's Chronometry paved a solid foundation) as well as to its possible (higher-dimensional) generalizations. The DLF theory (see [5]) can be understood as an attempt to modify the Standard Model of current theoretical physics by flexing the Poincare symmetry to certain 7-dimensional symmetries.

Sobolev Institute of Mathematics (Novosibirsk), Boston University (USA)
E-mail: `alevichev@gmail.com`

The D part of the theory is known as Segal's Chronometry (see [7, 8]) which is based on compact cosmos $D = U(2)$ with the $SU(2, 2)$ fractional linear conformal action on it.

2. Segal on Chronogeometry (from his book [8])

The text below is supplied with necessary references from Segal's book. These references are given at the end of the section.

2. Causality and geometry – historical

When Einstein questioned the absolute nature of simultaneity, and developed a theory of time and communication based on the propagation of light signals, causality considerations were implicitly introduced into the theory of space and time. These emerge more clearly in the work of Minkowski. However, causality was treated in a largely philosophical and intuitive way, as a marginal feature of quantitatively more central matters, e.g., the addition of large velocities. Indeed, the latter feature is identified by Bridgeman as the main one in relativity, and one that is logically unrelated to causality. None of these authors, nor their immediate successors, attempted an axiomatic treatment; nor made a consistent explicit separation between mathematical and physical considerations; and to this day (with the exceptions noted below), the notions of "causality," "observer," "clock," and "rod," are commonly used in quite intuitive, if not subjective, senses in work in relativity theory.

The explicit and cogent significance of causality for relativity theory was first recognized and emphasized by Robb (in 1911-1936) and developed by him into a deductive theory in which special relativity is effectively derived without any use of such notions as "clock" or "simultaneity" (at different points of space). As recognized by Fokker (1965), Robb thereby founded the subject of *chronogeometry*, in which considerations of temporal order are merged with geometry in a mathematical way, but with a presumption of applicability to physical space-time. A central notion in Robb's theory was a partial ordering in a given space, representing physically the relation of temporal precedence, in the world's space-time medium. Many mathematical axioms, in significant part motivated by optical considerations, with relatively objective physical interpretations (not requiring notions such as observer, clock, or rod at *different* points of space-time) lead after extensive analysis in this theory to the conclusion that the given causality-endowed space is isomorphic to Minkowski space-time, the partial ordering being the usual notion of temporal precedence in this manifold. By modern standards, while Robb's work was quite original and exhibits high order of mathematical clarity and coherence, it was isolated, unsophisticated, and apparently terminal in intent. Its main significance seems to lie in its formulation of the causality point of view, and demonstration of its power to lead to a more objective and at the same time philosophically satisfying treatment of relativity.

Since the war, the subject of chronogeometry has attracted the attention of a number of mathematicians, including notably A. D. Alexandrov and E. C. Zeeman, and in a modified form, J. L. Tits. In work beginning in the early 1950's, Alexandrov developed a school of work on mathematical relativity, very much from the chronogeometric outlook (explicitly so in Alexandrov, 1967, a key work), which has been contributed to by Busemann (1967) and Pimenov (1970), among others. In 1964 Zeeman rediscovered and exposed cogently the theorem (due originally to Alexandrov and Ovchinnikova, 1953, a work which seems not to have been widely disseminated outside the Soviet Union), that a causality-preserving transformation of Minkowski space is necessarily a Lorentz transformation, within a scalar factor. In 1960 Tits, in a key work, published a summary of a classification of all four-dimensional Lorentzian manifolds enjoying certain physically natural transitivity properties.

Chronogeometry has also emerged, in quite a different although related way, from the needs of the general theory of hyperbolic partial differential equations, and our initial acquaintance with the subject was derived from the fundamental work of Leray (1952), which correlated in a very general way the infinitesimal and finite notions of causality. A given hyperbolic equation defines an infinitesimal notion of temporal order, in the form of proper convex cone in the tangent space at each point of the space-time manifold. Prewar work by Zaremba and Marchand was completed and applied with cogency in Leray's work. His work, and particularly its chronometric side, has been further developed by Choquet-Bruhat (1971), who has made applications to general relativity; somewhat related work is due to Lichnerowicz (1971). Partially similar but more intricate and specialized ideas have been applied to the problem of the structure of space-time in general relativity by Hawking, Penrose, and a number of collaborators (cf. Ellis and Hawking, 1973), as well as by other recent writers on the problem of singularities in general relativity.

The subject of hyperbolic partial differential equations in the large can be considered in large part as falling under the general heading of causality and evolutionary considerations in functional analysis. This is not a question of pure geometry, of course, but rather of function spaces built on the space-time manifold; nevertheless there are some essential geometrical aspects, and causality plays a crucial role. This is the case, for example, for the key notions of domain dependence, region of influence, and of causal propagation. Indeed, hyperbolicity may well be necessary as well as sufficient for causal propagation, as evidenced in part by recent work of Berman (1974). This shows in particular that in the Klein-Gordon equation ($c = constant$), it is impossible to replace delta $+c$ by any other semi-bounded self-adjoint operator in L_2 over space if propagation is to remain both causal and Euclidean-invariant in Minkowski space.

The latter work continues an extensive line of work on the implications of causality for temporally invariant linear operators. The treatment of the dispersion of light by Kramers and Kronig was among the earliest and most influential in this general direction. The work of Bode on the design of wave filters applied a similar idea in a nonrelativistic context, that of linear network theory. Mathematically, the work of Paley and Wiener on complex Fourier analysis, and of Kolmogoroff, and later Wiener, and many others on linear prediction theory, in part relate to causality considerations in a context of temporal development and invariance. The Paley-Wiener theory was extended to a more general setting, applicable to relativistic cases, by Bochner. This was used in the postwar development of the general theory of linear hyperbolic equations due to Garding and Leray, and thereby connected with causality features.

A partial synthesis of the causality ideas involved in this line of work is involved in the abstract study of linear systems by Foures and Segal (1955). A general conclusion which is relevant to the present considerations and which emerges from this work is that the “future” may be represented by an essentially arbitrary nontrivial closed convex cone in the underlying linear manifold, without any fundamental loss of scientific cogency in the treatment of global questions. Furthermore, the convexity of the cone is both physically natural and technically crucial.

References

Alexandrov, A. D., and Ovchinnikova, V. V. (1953). *Vest. Leningrad gos. Univ.* **11**, 95.

Alexandrov, A. D. (1967). *Canad. J. Math.* **19**, 1119.

Berman, S. J. “Wave equations with finite time velocity of propagation.” Doctoral dissertation, MIT. Published: *Trans. Amer. Math. Soc.* **188** (1974), 127-148.

Busemann, H. (1967). *Dissertationes Math.* (*Rozprawy Mat.*) **53**, 52pp.

Choquet-Bruhat, Y. (1971). *General Relativity and Gravitation* **2**, 1.

Ellis, G. F. R., and Hawking, S. W. (1973). “The Large-Scale Structure of Space-Time.” Cambridge Univ. Press, London and New York.

Fokker, A. D. (1964). “Time and Space, Weight and Inertia.” Pergamon Press, Oxford.

Foures, Y., and Segal, I. (1955). *Trans. Amer. Math. Soc.* **78**, 385.

Leray, J. (1952). “Hyperbolic Partial Differential Equations.” Institute for Advanced Study, Princeton, New Jersey.

Lichnerowicz, A. (1971). *General Relativity and Gravitation* **1**, 235.

Pimenov, R. I. (1970). "Kinematic Spaces" (Seminars in Mathematics, Vol.6). Steklov Mathematics Institute, Leningrad; translated and published by Consultants Bureau, New York.

Robb, A. A. (1936). "Geometry of Space and Time." Cambridge Univ. Press, London and New York.

Tits, J. (1960). "Colloque sur la theorie de la relativite." Centre Belge Recherches Math., 107.

Zeeman, E. C. (1964). *J. math. Phys.* **5**, 490.

3. Embedding of $U(1,1)$ into $U(2)$ and generalizations to higher dimensions: the Sviderskiy formula

A.A. Akopyan. 2012

Let us start with a brief discussion of the general case. Given nonnegative integers p and q, we define the Lie algebra *u(p,q)* as the totality of all $p+q$ by $p+q$ matrices m (complex entries allowed) which satisfy

$$ms + sm^* = 0, \tag{3.1}$$

the above s is the diagonal matrix with p ones and q negative ones on the principal diagonal.

Formula

$$n = sm \tag{3.2a}$$

defines a linear bijection between vector spaces of Lie algebras *u(p,q)* and *u(p+q)*: (3.2a) is mentioned on p.219 of [2]. Obviously,

$$m = sn \tag{3.2b}$$

is the formula for the inverse mapping from *u(p+q)* onto *u(p,q)*.

Formulas (3.2a),(3.2b) might be viewed as giving canonical linear correspondence between *u(p,q)* and *u(p+q)* but how about correspondence between Lie groups *U(p,q)* and *U(p+q)*?

The research in this direction has been started (see [4]) by the second author together with late Oleg S. Sviderskiy (31 July 1969 – 30 March 2011). As a tribute to Oleg, it is now suggested that the formula for the canonical correspondence between groups *U(p,q)* and *U(p+q)* be known as the *Sviderskiy formula*; it is presented below as Theorem 1.

We define the Lie group *U(p,q)* as the totality of all $p+q$ by $p+q$ matrices Z (complex entries allowed) which satisfy

$$Z^* s Z = s \tag{3.3}$$

with s introduced above. We now describe how *U(1,1)* sits in *U(2)*. This is defined by the following function ***h*** from *D=U(2)*: the image of a matrix

$$Z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$$

from *U(2)* is the matrix V with

$$v_1 = d/z_4, v_2 = z_2/z_4, v_3 = -z_3/z_4, v_4 = 1/z_4, \tag{3.4}$$

here d is the determinant of Z. Notice that the determinant of V equals z_1/z_4.

Proposition 1. *The mapping* (3.4) *is only undefined for elements Z on the torus* $z_1 = z_4 = 0$ *in* $D = U(2)$. *The image is the entire F=U(1,1). In terms of Lorentzian metrics (introduced in* [5] *on both D and F) the mapping* (3.4) *is conformal. The tangent mapping (or the differential of* ***h****) at the neutral element of D is exactly our* (3.2b). ■

Here we only notice that correspondence (3.4) is similar to the one established in [5, Theorem 6] whereas other details of the **proof** are to be presented elsewhere.

Remark 1. Significant part of what is discussed in this section, also makes sense in the *SO(p,q)* vs *SO(p+q)*context.

We now proceed with the Sviderskiy formula which defines an embedding of *U(p,q)* into *U(p+q)* as manifolds. This mapping is defined as a fractional linear application of a certain *2n* by *2n* matrix W to (all) matrices in *U(p,q)*; here *n=p+q*. The n by n blocks A, B, C, D of the matrix W are defined as follows:

$$A = D = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = C = \begin{bmatrix} 0 & 0 \\ 0 & I_q \end{bmatrix},$$

where I_p (respectively, I_q) stand for the unit matrix of size p (respectively, of size q).

Theorem. (*The Sviderskiy formula*). *The fractional linear application of the above introduced matrix W is defined for all matrices in U(p,q), and U(p,q) is in a one-to-one correspondence with its image. The inverse mapping is also defined as the fractional linear transformation (by the same matrix W).*

The **proof** is to be presented elsewhere.

Remark 2. The above (3.4) is a special case of the Sviderskiy formula.

4. The projective world underlying $U(2)$

In [1] we notice that there is a 2-cover P of $S^1 \times SO(3)$ (we denote this group $D^{1/2}$) by the group $U(2)$: P sends a matrix z into a pair *(det z,v)*. The matrix v here is the image of u under a standard covering map p from $SU(2)$ onto $SO(3)$.

Finally, u (being a matrix from $SU(2)$) is determined (up to a sign) from the decomposition $z = du$, here $d^2 = det\ z$. Both P and p are group homomorphisms.

From the theoretical physics viewpoint, the fundamental role of p is well known. Regarding P, its mere existence allows to present Segal's chronometric theory (see [5, 7] and References therein) starting with the "lowest level" possible (since the center of the group $SO(3)$ is trivial). It is well known how the Lie group $U(2)$ (with a bi-invariant Lorentzian metric on it) can be viewed as a conformal compactification of the Minkowski spacetime. That gives one approach to how to study physics of Segal's compact cosmos $D = U(2)$. The Lie group $D^{1/2}$ inherits a bi-invariant metric through the above introduced homomorphism P and we call $D^{1/2}$ the *projective world*, it thus underlies Segal's compact cosmos $D = U(2)$. That gives us an option of a second way how to build physics in D: to start with that of $D^{1/2}$.

This last topic is to be discussed elsewhere, while in [1] we prove that the projective group of the projective space $SO(3)$ is the (component of) group $SO(3, 3)$. One would probably anticipate such an observation: it is known since long ago that a projective transformation between two projective lines is a fractional linear one (see [6, p.22]). In [1] we introduce the fractional linear action of $SO(3, 3)$ on $SO(3)$ and we prove that it is a (globally defined) projective action (from now and on, notation $SO(3, 3)$ stands for the connected component of the group which contains the unit matrix). The key ingredient of the proof is the commutative diagram which (essentially) intertwines the linear action of $SL(4)$ in $\mathbb{R}^4$ with the $SO(3, 3)$ fractional linear action on $SO(3)$.

5. Acknowledgments

As regards the first author, this project was funded [in part] by Boston University UROP. The authors are grateful to the Elsevier Ltd for the permission to reproduce excerpts (pp. 2 – 5) from [8].

References

1. Akopyan A.A., Levichev A.V. On $SO(3, 3)$ as the projective group of the space $SO(3)$ // submitted to the Siberian Advances in Mathematics.
2. Dubrovin B.A., Fomenko A.T., Novikov S.P. Modern Geometry — Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields. Springer, 1991.
3. Guts A.K. Chronogeometry. Axiomatic Relativity Theory. Omsk, OOO "UniPack", 2008 (in Russian)
4. Levichev A.V., Sviderskiy O.S. Lie groups *U(p,q)* of matrices as a single system based on fractional linear transformations: I. General consideration and cases $p+q=2, 3$. // In: Proceedings of the International Conference "Contemporary Problems of Analysis and Geometry". Novosibirsk: Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences. 2009. P. 68–69.

5. Levichev A.V. Pseudo-Hermitian realization of the Minkowski world through the DLF-theory // Physica Scripta. 2011. N. 1. P. 1–9.
6. Onishchik A.L., Sulanke R. Projective and Cayley-Klein Geometries. Springer-Verlag Berlin Heidelberg, 2006.
7. Paneitz S.M., Segal I.E. Analysis in space-time bundles I: General considerations and the scalar bundle // Journal of Functional Analysis. 1982. V.47. P. 78–142.
8. Segal I.E. Mathematical Cosmology and Extragalactic Astronomy. New York : Academic Press, 1976.