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IF ENERGY IS NOT PRESERVED, THEN PLANCK'S CONSTANT IS NO LONGER A CONSTANT: A THEOREM

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For any physical theory, to experimentally check its validity, we need to formulate an alternative theory and check whether the experimental results are consistent with the original theory or with an alternative theory. In particular, to check whether energy is preserved, it is necessary to formulate an alternative theory in which energy is not preserved. Formulating such a theory is not an easy task in quantum physics, where the usual Schroedinger equation implicitly assumes the existence of an energy (Hamiltonian) operator whose value is preserved. In this paper, we show that the only way to get a consistent quantum theory with energy non-conservation is to use Heisenberg representation in which operators representing physical quantities change in time. We prove that in this representation, energy is preserved if and only if Planck's constant remains a constant. Thus, an appropriate quantum analogue of a theory with non-preserved energy is a theory in which Planck's constant can change – i.e., is no longer a constant, but a new field.

1. Formulation of the Problem

Every physical law needs to be experimentally tested. Physics is a rapidly changing science, new discoveries are being made all the time, experimental discoveries that are often inconsistent with the existing physics and which lead to a development of new physical theories. Testing the existing physical theories is one of the main ways how physics evolves.

How physical laws can be experimentally tested. To test a physical law, we must:

- formulate an alternative theory in which this law is not valid (while others are valid),
- find a testable experimental situation in which the predictions of this alternative theory differ from the predictions of the original theory, and then
- experimentally check which of the two theories is correct.

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Example. This is how the General Relativity theory (alternative at that time) was experimentally tested: by experimentally comparing the predictions of Newton's gravitation theory – the prevalent theory of that time – with the predictions of the alternative theory; see, e.g., [3].

How can we test energy conservation law? A problem. One of the fundamental physical laws is the energy conservation law. At first glance, checking this law is easy: even on the level of Newton's physics, with the usual equations of motion

$$\frac{d^2x_i}{dt^2} = \frac{1}{m} \cdot f_i, \tag{1}$$

relating acceleration $\frac{d^2x_i}{dt^2}$ with the force f_i, there are many non-potential force fields $f_i(x)$ in which energy is not preserved.

The problem appears when we take quantum effects into account, i.e., when we consider the quantum equations. In quantum physics, the main equation – originally formulated by Schroedinger – has the form (see, e.g., [1])

$$\mathrm{i} \cdot \hbar \cdot \frac{\partial \psi}{\partial t} = H\psi, \tag{2}$$

where $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$, $\psi(x)$ is the wave function describing the quantum state, and H is a so-called *Hamiltonian*, an operator describing the energy of a state. In a non-potential force field, there is no well-defined notion of a total energy and thus, it is not possible to write down the corresponding quantum equation.

Discussion. The need to test the energy conservation law on quantum level is not purely theoretical:

- on a pragmatic level, serious physicists considered the possibility of micro-violations of energy conservation starting from the 1920s [1];
- on a more foundational level, the intuitive ideas of free will seem to lead to possible energy non-conservation [2].

What we do in this paper. In this paper, we show how to form a quantum theory in which energy is not conserved. Specifically, we show that for quantum theories, energy non-conservation is equivalent to changing Planck's constant. Thus, in quantum physics, checking whether energy is conserved is equivalent to checking whether Planck's constant changes.

2. Analysis of the Problem

Schroedinger and Heisenberg representations of quantum physics: reminder. In quantum physics (see, e.g., [1]), states are described by elements of a Hilbert space – e.g., of the space of all square-integrable functions $\psi(x)$ – and physical quantities are described by linear operators in this space.

Historically, quantum physics started with a description by Heisenberg, in which states are fixed but operators change. Very soon, it turned out that in most cases, an alternative representation is more computationally advantageous – a representation in which operators are fixed but states change. This representation was originally proposed by E. Schroedinger and is therefore known as the Schroedinger representation.

Since we cannot use Schroedinger's representation, we will use the Heisenberg one. As we have mentioned, the Schroedinger's equation implicitly assumes the existence of (preserved) energy. Thus, to describe situations in which energy is not preserved, it is reasonable to use the Heisenberg representation.

Heisenberg representation: first approximation. In the Heisenberg representation, physical quantities like coordinates x_i and components $p_i \stackrel{\text{def}}{=} m \cdot \dfrac{dx_i}{dt}$ of the momentum vector are represented by operators. In the first approximation, the usual quantum mechanics is described by the usual Newton's equations

$$\frac{dx_i}{dt} = \frac{1}{m} \cdot p_i, \quad \frac{dp_i}{dt} = f_i, \tag{3}$$

with the only difference that instead of scalars x_i and p_i, we now consider operators. This description was first found by P. Ehrenfest (see, e.g., [1]).

The difference between the scalars and operators is that operators, in general, do not commute, i.e., in general, for two operators a and b, we have $[a, b] \stackrel{\text{def}}{=} ab - ba \neq 0$. Specifically, in the usual quantum physics, operators x_i and x_j corresponding to different coordinates commute with each other, operators p_i and p_j commute with each other, but operators x_i and p_i do not commute:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [p_i, x_j] = \mathrm{i} \cdot \hbar \cdot \delta_{ij}, \tag{4}$$

where the Kronecker's delta δ_{ij} is equal to 1 when $i = j$ and to 0 otherwise. For the usual energy-preserving quantum mechanics, these commuting relations get preserved as the operators x_i and p_i change in time – in accordance with Ehrenfest equations (3).

In the first approximation, the commutator $[a, b]$ can be described in terms of the Poisson brackets (see, e.g., [1]). Namely, for arbitrary functions $a(x, p)$ and $b(x, p)$ of coordinates $x = (x_1, x_2, x_2)$ and momentum $p = (p_1, p_2, p_3)$, we have

$$[a, b] = \mathrm{i} \cdot \hbar \cdot \{a, b\} + o(\hbar), \tag{5}$$

where

$$\{a, b\} \stackrel{\text{def}}{=} \sum_k \left(\frac{\partial a}{\partial p_k} \cdot \frac{\partial b}{\partial x_k} - \frac{\partial a}{\partial x_k} \cdot \frac{\partial b}{\partial p_k} \right). \tag{6}$$

As an example, let us show what happens for the Heisenberg commutator $[a,b]$ for which $a=p_i$ and $b=x_j$. Since $a=p_i$ depends only on p_i, we have $\dfrac{\partial p_i}{\partial p_k}=\delta_{ik}$ and $\dfrac{\partial p_i}{\partial x_k}=0$. Similarly, since $b=x_j$ depends only on x_j, we have $\dfrac{\partial x_j}{\partial p_k}=0$ and $\dfrac{\partial x_j}{\partial x_k}=\delta_{jk}$. Thus,

$$\{p_i,x_j\}=\sum_k \delta_{ik}\cdot\delta_{jk}=\delta_{ij}.$$

When the force comes from a potential field, Planck's constant is preserved. Let us show that in the potential field with potential energy $V(x)$, when $f_i=-\dfrac{\partial V}{\partial x_i}$, Planck's constant is preserved. Indeed, let us assume that the commuting relations (4) hold at a certain moment of time t_0. In particular, this means that $[p_i,x_j]=\mathrm{i}\cdot\hbar\cdot\delta_{ij}$. Let us show that – at least in the first approximation – this relation is preserved, in the sense that $\dfrac{d}{dt}[p_i,x_j]=0$.

First, we should note that since $[a,b]=ab-ba$, we have

$$\frac{d}{dt}([a,b])=\frac{d}{dt}(ab-ba)=\frac{da}{dt}b+a\frac{db}{dt}-\frac{db}{dt}a-b\frac{da}{dt}=\left[\frac{da}{dt},b\right]+\left[a,\frac{db}{dt}\right].$$

Thus, we have

$$\frac{d}{dt}([p_i,x_j])=\left[\frac{dp_i}{dt},x_j\right]+\left[p_i,\frac{dx_j}{dt}\right]$$

Due to Ehrenfest equations, we have $\dfrac{dp_i}{dt}=f_i$ and $\dfrac{dx_i}{dt}=\dfrac{1}{m}\cdot p_i$, so we have

$$\frac{d}{dt}([p_i,x_j])=[f_i,x_j]+\frac{1}{m}\cdot[p_i,p_j]. \tag{7}$$

Since f_i depend only on the coordinates, and all coordinate operators commute, we have $[f_i,x_j]=0$. Since all the components of the momentum commute, we have $[p_i,p_j]=0$. Thus, we conclude that $\dfrac{d}{dt}([p_i,x_j])=0$.

Similarly, we can conclude that the second derivative of the Heisenberg commutator is also equal to 0. Indeed, by differentiating both sides of the equation (7), we conclude that

$$\frac{d^2}{dt^2}([p_i,x_j])=\left[\frac{df_i}{dt},x_j\right]+\frac{1}{m}\cdot[f_i,p_j]+\frac{1}{m}\cdot[f_i,p_j]+\frac{1}{m}\cdot[p_i,f_j]. \tag{8}$$

Here, since f_i depends only on coordinates, we have

$$\frac{df_i}{dt}=\sum_\ell\frac{\partial f_i}{\partial x_\ell}\cdot\frac{dx_\ell}{dt}=\frac{1}{m}\cdot\sum_\ell\frac{\partial f_i}{\partial x_\ell}\cdot p_\ell,$$

so
$$\frac{d^2}{dt^2}([p_i,x_j])=\frac{1}{m}\cdot\left(\left[\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell, x_j\right]+2[f_i,p_j]+[p_i,f_j]\right).$$
Thus, to prove that this second derivative is equal to 0, it is sufficient to prove that the expression in parentheses is equal to 0. In the first approximation, this expression is proportional to the sum S of the corresponding Poisson brackets
$$S=\left\{\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell, x_j\right\}+2\{f_i,p_j\}+\{p_i,f_j\};$$
so, in the first approximation, it is sufficient to prove that the sum S is equal to 0. In the first bracket, x_j depends only on x_j, so
$$\left\{\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot p_\ell, x_j\right\}=\sum_k\sum_\ell \frac{\partial f_i}{\partial x_\ell}\cdot\delta_{k\ell}\cdot\delta_{jk}=\frac{\partial f_i}{\partial x_j}.$$
For the second term of the sum S, since p_j only depends on the momentum, we get
$$\{f_i,p_j\}=-\sum_k \frac{\partial f_i}{\partial x_k}\cdot\delta_{jk}=-\frac{\partial f_i}{\partial x_j}.$$
Similarly,
$$\{p_i,f_j\}=\frac{\partial f_j}{\partial x_i}.$$
Thus, we have
$$S=\frac{\partial f_j}{\partial x_i}-\frac{\partial f_i}{\partial x_j}.$$
For the potential field, we have $f_i=-\dfrac{\partial V}{\partial x_i}$ and therefore,
$$\frac{\partial f_i}{\partial x_j}=-\frac{\partial^2 V}{\partial x_i\partial x_j}.$$
Hence, we have
$$S=-\frac{\partial^2 V}{\partial x_i\partial x_j}+\frac{\partial^2 V}{\partial x_i\partial x_j}=0.$$

What happens when energy is not preserved: an example. We are interested in situations where energy is not preserved. Let us start our analysis with the simplest such situation of the friction force $f_i=-k\cdot v_i$, i.e., force of the type $f_i=-k_0\cdot p_i$, where $k_0\stackrel{\text{def}}{=}\dfrac{k}{m}$. In this case, from the formula
$$\frac{d}{dt}([p_i,x_j])=[f_i,x_j]+\frac{1}{m}\cdot[p_i,p_j],$$

by using $[p_i, p_j] = 0$, we get

$$\frac{d}{dt}([p_i, x_j]) = -k_0 \cdot [p_i, x_j].$$

In other words, for $h \stackrel{\text{def}}{=} [p_i, x_i]$, we have a differential equation

$$\frac{dh}{dt} = -k_0 \cdot h.$$

From this equation, we conclude that $\frac{dh}{h} = -k_0 \cdot dt$ hence $\ln(h) = \text{const} - k_0 \cdot t$, and $h(t) = \text{const} \cdot \exp(-k_0 \cdot t)$. We know that $h(t_0) = \mathrm{i} \cdot \hbar$, hence

$$h(t) = h(t_0) \cdot \exp(-k_0 \cdot (t - t_0)).$$

At the initial moment t_0, we have $h(t_0) = \mathrm{i} \cdot \hbar$. So, the above equation means, in effect, that Planck's constant $\hbar \stackrel{\text{def}}{=} \frac{[p_i, x_i]}{\mathrm{i}}$ is no longer a constant – it exponentially decreases with time.

Discussion. Let us show that the same phenomenon – of Planck's constant no longer being a constant – occurs for every theory in which energy is not preserved.

3. Main Result

Formulation of the main result. Let us consider the general case, when each component f_i of a force is a function of coordinates x and momentum p. We will show that if in the quantum version of this theory, Planck's constant remains a constant, i.e., we have $[p_i, x_j] = \mathrm{i} \cdot \hbar \cdot \delta_{ij}$ for all moments of time, then the field f_i is a potential field, i.e., has the form $f_i = -\frac{\partial V}{\partial x_i}$ for some function $V(x)$.

This means that if f_i is *not* a potential field, then Planck's constant is no longer a constant.

Proof. If Planck's constant is a constant, this means, in particular, that we have $\frac{d}{dt}([p_i, x_j]) = 0$. Explicitly differentiating the left-hand side, we conclude that $[f_i, x_j] + \frac{1}{m} \cdot [p_i, p_j] = 0$. Since $[p_i, p_j] = 0$, we get $[f_i, x_j] = 0$. In the first approximation, this means that the corresponding Poisson bracket is equal to 0: $\{f_i, x_j\} = 0$. Since x_j depends only on the coordinate, we get

$$\{f_i, x_j\} = \sum_k \frac{\partial f_i}{\partial p_k} \cdot \delta_{kj} = \frac{\partial f_i}{\partial p_j} = 0.$$

The fact that all partial derivatives of f_i relative to p_j are equal to 0 means that f_i does not depend on the momentum. In other words, the force f_i depends only on the coordinates x.

Now, since we know that f_i depend only on the coordinates, for the second time derivative $\frac{d^2}{dt^2}([p_i, x_j])$, we can repeat arguments from the previous section and conclude that in the first approximation, this second derivative is proportional to

$$S = 2 \cdot \frac{\partial f_i}{\partial x_j} - 2 \cdot \frac{\partial f_i}{\partial x_j}.$$

So, from the fact that the second derivative is equal to 0, we conclude that $S = 0$, i.e., that

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i}{\partial x_j}$$

for all i and j. It is known that these equalities are necessary and sufficient conditions for the existence of a field V for which $f_i = -\frac{\partial V}{\partial x_i}$. Thus, we have proved that f_i is indeed a potential field. ■

Discussion. Theories in which Planck's constant is no longer a constant but a new physical field $s(x)$ have been proposed; see, e.g., [4].

It should be mentioned that we may need to go beyond proposed theory: indeed, these theories only consider a *scalar* field $s(x)$ corresponding to

$$[p_i, x_j] = \mathrm{i} \cdot \hbar \cdot s(x) \cdot \delta_{ij},$$

while, in general, the commutator $[p_i, x_j]$ can be an arbitrary *tensor*.

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