Mathematical Structures and Modeling
2013. N. 1(27). PP. 38–41

UDC 519.6

FOR DESCRIBING UNCERTAINTY, ELLIPSOIDS ARE BETTER THAN GENERIC POLYHEDRA AND PROBABLY BETTER THAN BOXES: A REMARK

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For a single quantity, the set of all possible values is usually an interval. An interval is easy to represent in a computer: e.g., we can store its two endpoints. For several quantities, the set of possible values may have an arbitrary shape. An exact description of this shape requires infinitely many parameters, so in a computer, we have to use a finite-parametric approximation family of sets. One of the widely used methods for selecting such a family is to pick a symmetric convex set and to use its images under all linear transformations. If we pick a unit ball, we end up with ellipsoids; if we pick a unit cube, we end up with boxes and parallelepipeds; we can also pick a polyhedron. In this paper, we show that ellipsoids lead to better approximations of actual sets than generic polyhedra; we also show that, under a reasonable conjecture, ellipsoids are better approximators than boxes.

1. Formulation of the Problem

Need for describing sets of possible values. Measurement and estimates are never 100% accurate. As a result, we usually do not know the exact value of a physical quantity; we usually know the set of possible values of this quantity. For a single quantity, this set is usually an interval. Representing an interval in a computer is easy: e.g., we can represent an interval by its endpoints; see, e.g., [7, 10].

For several quantities $x_1, \ldots, x_n$, in addition to interval bounds on each of these quantities, we often have additional restrictions on their combinations; as a result, the set of possible values of $x = (x_1, \ldots, x_n)$ can have different shapes. The space of all possible sets is infinite-dimensional, meaning that we need infinitely many real-valued parameters to represent a generic set. In a computer, at any given time, we can only store finitely many parameters; so, we cannot represent generic sets exactly, we need to approximate them by sets from a finite-parametric family.

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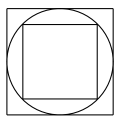
Convex set-based representation of sets of possible values. In many practical situations, e.g., when x_i are spatial coordinates, the selection of the quantities is rather arbitrary: we can use a different coordinate system in which, instead of the original quantities x_i, we use linear combinations $y = Tx$, i.e., $y_i = \sum\limits_{j=1}^{m} t_{ij} \cdot x_j$. In view of this, a reasonable way to select a finite-parametric set is to pick a bounded symmetric convex set S_0 with non-empty interior, and to use images TS_0 of this set S_0 under arbitrary linear transformations T.

If we start with a Euclidean unit ball $S_0 = B \stackrel{\text{def}}{=} \left\{x : \sum\limits_{i=1}^{n} x_i^2 \le 1\right\}$, we get the family of ellipsoids (see, e.g., [1–4, 11–14, 16]); if we start with a unit cube $S_0 = C \stackrel{\text{def}}{=} \{x : |x_i| \le 1 \text{ for all } i\}$, we get the family of all boxes (plus the corresponding parallelepipeds); alternatively, we can also start with a symmetric convex polyhedron P.

Which set S_0 should we choose? Once we pick a set S_0, we can (precisely) represent sets S of the type TS_0. If we start with such a set S, we enclose it into a set $TS_0 = S$, and then, if we want to enclose TS_0 in a set $\lambda \cdot S$ corresponding to the original S-based representations, we get the same original set $S = TS_0$ back, with $\lambda = 1$.

For sets S which are different from TS_0, the S_0-based representation is only approximate. We start with a set S, and we enclose it in a set $TS_0 \supseteq S$ for an appropriate linear transformation T. If we then try to enclose TS_0 in a set of the type $\lambda \cdot S$, then we inevitably get $\lambda > 1$.



The smaller λ, the better the approximation. It is therefore reasonable, as a measure $d(S_0, S)$ of accuracy of approximating S by S_0, to use the smallest λ corresponding to all possible T:

$$d(S_0, S) = \inf\{\lambda : \exists T\, (S \subseteq TS_0 \subseteq \lambda \cdot S)\}.$$

This quantity is known as a *Banach-Mazur distance* between the convex sets S and S_0; see, e.g., [15, 17].

For each "standard" set S_0, we get different values $A(S_0, S)$ for different sets S. As a measure of quality $Q(S_0)$ of choosing S_0, it is reasonable to select the worst-case approximation accuracy

$$Q(S_0) \stackrel{\text{def}}{=} \sup_S d(S_0, S),$$

where the supremum is taken over all possible bounded symmetric convex sets S with non-empty interior.

2. Main Results

Main conclusion: ellipsoids are better than generic polyhedra. According to the well-known John's Theorem [8, 15, 17], for the Euclidean unit ball B, we have $d(B, S) \le \sqrt{n}$ for all symmetric convex sets S. Thus, we have $Q(B) \le \sqrt{n}$.

On the other hand, according to Gluskin's theorem [6, 15, 17], there exists a constant $c > 0$ such that for each dimension n, there exist polyhedra P and P' for which $d(P, P') \ge c \cdot n$ and for which, therefore, $d(P) \ge c \cdot n$. Moreover, if we take a convex hull P of $2n$ points randomly selected from a unit Euclidean sphere, then, with high probability, we get $Q(P) \ge c \cdot n$. Since for large n, we have $c \cdot n \gg \sqrt{n}$ and therefore, $Q(B) \ll Q(P)$, this shows that for large dimensions, ellipsoids are indeed better than generic polyhedra.

Additional conclusion: ellipsoids are probably better than boxes. A Euclidean unit ball B (corresponding to ellipsoids) and a unit cube C (corresponding to boxes) can be viewed as particular cases of unit balls $B_p \stackrel{\text{def}}{=} \{x : \|x\|_p \le 1\}$ in the ℓ_p-metric $\|x\|_p \stackrel{\text{def}}{=} \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{1/p}$: B is a unit ball in the ℓ_2-metric while C is a unit ball in the ℓ_∞-metric: $B = B_2$ and $C = B_\infty$. The exact values of $d(B_p, B_q)$ are known only when both p and q are on the same side of 2; in this case, $d(B_p, B_q) = n^{|1/p - 1/q|}$. In particular, for $p = 1$ and $q = 2$, we get $d(B_1, B_2) = \sqrt{n}$.

These values have the property that when $p < q$, then $d(B_p, B_q)$ strictly increases when p decreases or when q increases; in other words, the larger the difference between p and q, the larger the value $d(B_p, B_q)$. For values p and q on different sides of 2, this monotonicity does not hold for $n = 2$, since in this case, B_1 (rhombus) and B_∞ (square) are linearly equivalent and thus, $d(B_1, B_\infty) = 0$. However, for $n > 3$, we do not have this anomaly and therefore, it is reasonable to conjecture that for $n > 3$, this monotonicity holds. Under this hypothesis, $d(B_\infty, B_1) > d(B_2, B_1) = \sqrt{n}$, and thus, $Q(B_\infty) \ge d(B_\infty, B_1) > \sqrt{n}$. Since $Q(B_2) = \sqrt{n}$, we therefore conclude that $Q(B_2) < Q(B_\infty)$ and thus, ellipsoids are better than boxes.

Comment. These results are in line with a general result according to which, under certain conditions, ellipsoids are the best approximators [5, 9].

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and by a grant on F-transforms from the Office of Naval Research.

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