Mathematical Structures and Modeling
2013. N. 2(28). PP. 15–20

UDC 514.82

STOCHASTIC CAUSALITY IS INCONSISTENT WITH THE LORENTZ GROUP

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According to modern physics, all physical processes are described by quantum theory. In particular, due to quantum fluctuations, even in the empty space, the causal relation is, in general, slightly different from the usual Minkowski one. Since quantum effects are probabilistic, to properly represent the corresponding stochastic causality, we need to describe, for every two events e and e', the probability $p(e, e')$ that e can causally influence e'. Surprisingly, it turns out that such a probability functions cannot be Lorentz-invariant. In other words, once we take into account quantum effects in causality, Lorentz-invariance is violated – similarly to the fact that it is violated if we take into account the presence of matter and start considering cosmological solutions.

1. Introduction

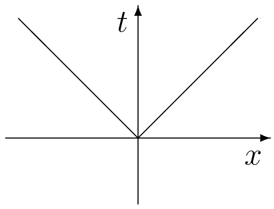
Deterministic causality in Special Relativity: a brief reminder. In the Minkowski space-time of Special Relativity, causality is deterministic: an event $e = (t, x)$ can causally influence an event $e' = (t', x')$ if and only if $t \le t'$, and a process with velocity not exceeding the speed of light c starting at e can reach e', i.e., if and only if

$$t' - t \ge \frac{d(x, x')}{c}, \tag{1}$$

where

$$d(x, x') \stackrel{\text{def}}{=} \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2} \tag{2}$$

denotes the Euclidean distance between the spatial points $x = (x_1, x_2, x_3)$ and $x' = (x'_1, x'_2, x'_3)$. Geometrically, this causality relation is described by the usual future cone:



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Deterministic causality implies Lorentz group. Special Relativity theory has many symmetries: namely, all physical phenomena (including the phenomenon of causality) remain the same if we simply perform a shift, a spatial rotation, and/or a Lorentz transformation

$$t' = \frac{t' - \frac{v \cdot x_1}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad x'_1 = \frac{x_1 - v \cdot t}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad x'_2 = x_2; \quad x'_3 = x_3. \tag{3}$$

It is known that, vice versa, the deterministic causality relation (1) *implies* Lorentz group, in the sense that every bijection $\mathbb{R}^4 \to \mathbb{R}^4$ which preserves this causality relation is a composition of shifts, spatial rotations, scaling $x \to \lambda \cdot x$, and a transformation from the Lorentz group; see, e.g., [1, 2, 5].

Deterministic causality beyond Special relativity. According to modern physics, the presence of matter changes the geometry of space-time; see, e.g., [4]. The resulting space-time is described by a *metric tensor* field $g^{ij}(x)$. This tensor field defined the following local causality relation: an event with 4-coordinates $e = (x_0, x_1, x_2, x_3)$ (where x_0 is time) can causally influence an event

$$e + de = (x_0 + dx_0, x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$$

if and only if

$$dx_0 \geq 0 \text{ and } \sum_{i=0}^{3} \sum_{j=0}^{3} g^{ij} \cdot dx_i \cdot dx_i \geq 0. \tag{4}$$

The space-time of Special relativity corresponds to the constant diagonal metric tensor $g^{ij} = \mathrm{diag}(c^2, -1, -1, -1)$.

Need for stochastic causality. According to modern physics, the real world is described by quantum theories; see, e.g., [2]. In quantum physics, there are always quantum fluctuations. In particular, even in the absence of matter, there are always quantum fluctuations of the metric tensor g^{ij}. As a result of these fluctuations, the values of $g^{ij}(x)$ are slightly different from their Special Relativity values $g^{ij} = \mathrm{diag}(c^2, -1, -1, -1)$. The resulting future cone may be slightly different from the original future cone (1):

- it may be that some events e' which are slightly outside the original future cone (1) can actually be influenced by the event e;
- it may also happen that some events e' on the border of the original future cone (1) cannot be influenced by the event e.

For example, if at some point x the value $g^{00}(x)$ is slightly larger than c^2 while other values $g^{ij}(x)$ remain the same, this means that we can have processes which are slightly faster than the original value of the speed of light, and thus some previously causally inaccessible events will be covered. Such microscopic local "violations" of causality are a known feature of quantum field theories [2, 4]; it should be mentioned that they do not lead to any violations of observed macroscopic causality.

Quantum fluctuations are random. As a result, we cannot tell beforehand which events can influence each other and which cannot. Instead, we can only talk about the *probability* $p(e, e')$ that an event e can influence an event e'.

Reasonable expectations. In the deterministic (non-quantum) case, to describe causality relation of Special Relativity, it is sufficient to write down the formula (1). Once we take into account the quantum-induced stochastic nature of causality, we need to describe a function $p(e, e')$. We are still discussing the space-time of Special Relativity, so it is reasonable to require that the corresponding function $p(e, e')$ does not change under Lorentz transformations.

What are other reasonable properties of the function $p(e, e')$? The probabilities $p(e, e')$ describe the results of quantum fluctuations. From the macroscopic viewpoint, quantum fluctuations are extremely small. We therefore expect that they only change causality relation in the very narrow vicinity of the border of the future cone, i.e., of the set of all the events (t', x') for which $t' = t + \frac{d(x, x')}{c}$. In other words:

- for events e' inside the future cone which are sufficiently far away from the border, we should have $p(e, e') \approx 1$;
- for events e' outside the future cone which are sufficiently far away from the border, we should have $p(e, e') \approx 0$;
- and only for the events in the narrow vicinity of the border, we should have values $p(e, e')$ which continually change from 0 to 1.

For a point e' on the original border of the future cone, it seems to be equally probable that the randomly perturbed causality cone will be perturbed towards the inside — in which case e can no longer influence e' — or it will be perturbed outside, in which case e can still casually influence e'. In other words, for such events e', we expect $p(e, e') \approx 0.5$.

On the other hand, if we take a symmetric point $e'' = e - (e' - e)$, we end up with a point $e'' = (t'', x'')$ on the *past* cone of e, for which $t'' = t - \frac{d(x, x'')}{c}$. This point e'' is far way from the border of the future cone, so we expect $p(e, e'') \approx 0$.

Since $p(e, e') \approx 0.5$ and $p(e, e'') \approx 0$, we conclude that for these three events e, e', and e'', we get $p(e, e') \gg p(e, e'')$. This should be true when e' (and hence e'') is sufficiently far away from e, i.e., this should be true for at least some triples of events.

What we show in this paper. In this paper, we prove, somewhat unexpectedly, that no such Lorentz-invariant function $p(e, e')$ is possible.

2. Main Result

Definition 1. *By* stochastic causality*, we mean a continuous function* $p : \mathbb{R}^4 \times \mathbb{R}^4 \to [0,1]$ *for which, for some point* e' *on the border of the future cone of* e*, we have* $p(e,e') > p(e,e'')$*, where* $e'' = e - (e' - e)$ *is the symmetric point on the border of the past cone of* e*.*

Definition 2. *We say that a stochastic causality function is* Lorentz-invariant *if* $p(Te,Te') = p(e,e')$ *for each Lorentz transformation* T *and for all possible events* e *and* e'*.*

Proposition. *Stochastic causality cannot be Lorentz-invariant.*

Comment. For readers' convenience, the proof of this result is placed in a separate section.

Discussion. What is the physical meaning of our result? This result shows that once we take into account *quantum effects* in causality, Lorentz-invariance is violated.

Lorentz-invariance is definitely violated when we take *matter* into account, since then instead of the flat Minkowski space-time, we have a cosmological space-time, and cosmological space-times are not Lorentz-invariant, they usually have a fixed temporal axis. We show that a similar phenomenon occurs even in the absence of matter, when we only take into account quantum fluctuations.

3. Proof of the Main Result

By definition of stochastic causality, there exist events e and e' for which e' is located on the border of the (Minkowski-based) future cone of e and for which $p(e,e') > p(e,e'')$, where $e'' \stackrel{\text{def}}{=} e - (e' - e)$.

For convenience, let us start with any coordinate system on the Minkowski space. Then, let us shift the starting point to the point e. In the new coordinates, the event e has coordinates $(0,0,0,0)$. Let $t \stackrel{\text{def}}{=} e'_0 - e_0 > 0$ denote the time coordinate of the difference $e' - e$; then, in the new coordinate system, the x_0-component of the event e' is equal to $e'_0 = t$.

Finally, let us rotate the spatial axes in such a way that x_1-axis goes in the direction of the spatial component of the different $e' - e$. Then, we will have $e'_2 = e'_3 = 0$. Since the point e' is located on the border of the future cone of $e = 0 = (0,0,0,0)$, we conclude that $e'_1 = c \cdot t$. Thus, $e' = (t, c \cdot t, 0, 0)$. Therefore, for the symmetric point $e'' = e - (e' - e)$, we get $e'' = (-t, c \cdot t, 0, 0)$.

We have $p(e,e') > p(e,e'')$, i.e.,

$$p(0,(t,c \cdot t,0,0)) > p(0,(-t,c \cdot t,0,0)). \tag{5}$$

Since the function $p(e, e')$ is continuous, for sufficiently small $\varepsilon > 0$, we have a similar inequality for ε-close points:

$$p(0, (t \cdot (1-\varepsilon), c \cdot t, 0, 0)) > p(0, (-t \cdot (1-\varepsilon), c \cdot t, 0, 0)). \quad (6)$$

We will show that this inequality is inconsistent with Lorentz-invariance; specifically, we will show that there exists a Lorentz transformation T that keeps the point $e = 0$ invariant and transforms $e' = (t \cdot (1-\varepsilon), c \cdot t, 0, 0)$ into $e'' = (-t \cdot (1-\varepsilon), c \cdot t, 0, 0)$. Thus, Lorentz-invariance would imply that

$$p(e, e') = p(0, (t \cdot (1-\varepsilon), c \cdot t, 0, 0)) = p(0, (-t \cdot (1-\varepsilon), c \cdot t, 0, 0)) = p(e, e''), \quad (7)$$

which contradicts to the inequality (6).

To find the proper Lorentz transformation (3), we should have

$$e''_0 = \frac{e'_0 - \dfrac{v \cdot e'_1}{c}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \quad (8)$$

and

$$e''_1 = \frac{e'_1 - v \cdot e'_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}. \quad (9)$$

Substituting $e''_0 = -t \cdot (1-\varepsilon)$, $e'_0 = t \cdot (1-\varepsilon)$ and $e'_1 = c \cdot t$ into the formula (8) and multiplying both sides by the denominator, we get

$$-t \cdot (1-\varepsilon) \cdot \sqrt{1 - \frac{v^2}{c^2}} = t \cdot (1-\varepsilon) - \frac{v}{c} \cdot t. \quad (10)$$

Dividing both sides of this formula (10) by t and denoting $u \stackrel{\text{def}}{=} \dfrac{v}{c}$, we get

$$-(1-\varepsilon) \cdot \sqrt{1-u^2} = (1-\varepsilon) - u. \quad (11)$$

Squaring both sides, we get

$$(1-\varepsilon)^2 \cdot (1-u^2) = (1-\varepsilon)^2 + u^2 - 2u \cdot (1-\varepsilon). \quad (12)$$

The left-hand side of (12) can be represented as $(1-\varepsilon)^2 - u^2 \cdot (1-\varepsilon)^2 \cdot u^2$. Cancelling terms $(1-\varepsilon)^2$ in both sides, we get

$$-(1-\varepsilon)^2 \cdot u^2 = u^2 - 2u \cdot (1-\varepsilon). \quad (13)$$

Moving terms depending on u^2 to the right-hand side and all the other terms to the left-hand side, we get

$$2u \cdot (1-\varepsilon) = [1 + (1-\varepsilon)^2] \cdot u^2. \quad (14)$$

Dividing both sides by u, we now get

$$u = \frac{2 \cdot (1 - \varepsilon)}{1 + (1 - \varepsilon)^2}. \tag{15}$$

One can show that for the corresponding velocity $v = u \cdot c$, the equality (9) is also satisfied. Indeed, substituting $e_1'' = c \cdot t$, $e_0' = t \cdot (1 - \varepsilon)$ and $e_1' = c \cdot t$ into the formula (9) and multiplying both sides by the denominator, we get

$$c \cdot t \cdot \sqrt{1 - \frac{v^2}{c^2}} = c \cdot t - v \cdot t \cdot (1 - \varepsilon). \tag{16}$$

Dividing both sides of (16) by $c \cdot t$ and using the notation $u = \frac{v}{c}$, we get

$$\sqrt{1 - u^2} = 1 - u \cdot (1 - \varepsilon). \tag{17}$$

Squaring both sides of (17), we get

$$1 - u^2 = 1 + u^2 \cdot (1 - \varepsilon)^2 - 2 \cdot u \cdot (1 - \varepsilon). \tag{18}$$

Subtracting 1 from both sides of this equality (18), moving terms depending on u^2 to the right-hand side and all the other terms to the left-hand side, we get the same formula (14) that was used to find u. Thus, for the value u described by the formula (15), both formulas (8) and (9) are satisfied. The proposition is proven.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and by a grant N62909-12-1-7039 from the Office of Naval Research.

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