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FROM URYSOHN'S UNIVERSAL METRIC SPACE TO A UNIVERSAL SPACE-TIME

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A known Urysohn's result shows that there exists a *universal* metric space, i.e., a metric space into every other (separable) metric space can be isomorphically embedded. Moreover, this universal metric space can be selected to be *ultra-homogeneous* — every isomorphism of its two finite subsets can be extended to the isomorphism of the whole space.
Starting with Einstein's theories of Special and General relativity, space-times are described by a different type of structure — a set (of events) equipped with the proper time $\tau(a, b)$ between points a and b; such spaces are known as *space-times with kinematic metric*, or *k-space-times*. In this paper, we show that Urysohn's result can be extended to k-space-times — namely, that there exists an ultra-homogeneous universal k-space-time.

1. Formulation of the Problem

Urysohn's universal metric space. In one of his latest papers [16], P. Urysohn showed that there exists a *universal* metric space U into which every separable metric space can be isometrically embedded; see, e.g., [1, 9, 10, 14]. The Urysohn's space is known to be *ultra-homogeneous* in the sense that every isomorphism between its two finite subspaces can be extended to an *auto-isometry*, i.e., an isometric 1-1 mapping of the space U onto itself.

Possible physical interpretation of Urysohn's universal metric space. From the physical viewpoint, Urysohn's universal set can be viewed as a universal *space* — a space that includes all other metric spaces as its subsets.

Need to consider space-times, not metric spaces. In Newton's physics, spatial distance $d(a, b)$ was a well-defined physical notion. Relativity theory has shown, however, that spatial distance is not absolute, it depends on the coordinate

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system; to properly describe the physical world, we thus need to consider *space-times* (sets of events), not just spaces; see, e.g., [2, 2, 5].

Proper time: a natural space-time analog of a metric. In relativity theory, the only "absolute" (invariant) analog of metric distance is the *proper time* $\tau(a,b)$ between two events a and b.

In relativity theory, the observed time interval depends on the person's motion: in general, the faster you travel, the shorter the observed time interval, so that for objects moving close to the speed of light, time practically stands still; this is the main idea behind the well-known "twins paradox". Thus, if an event a causally precedes event b, it is always possible to get from a to b by spending as little proper time as possible: it is sufficient to travel with a speed close to the speed of light, and then travel back with the same speed.

The longest possible proper time is when we are immobile (or when we move with constant direction in the same velocity); any acceleration decreases the amount of proper time. Thus, a physically meaningful proper time $\tau(a,b)$ can be defined as the longest possible proper time for all trajectories leading from the event a to the event b. When a is not causally preceding b, we can set $\tau(a,b) = 0$.

Comment. The fact that we consider the *longest* possible proper time make this notion different from (kind of dual to) the usual spatial metric $d(a,b)$ which can be defined as the length of the *shortest* possible path connecting spatial points a and b.

Properties of proper time. Clearly, we can get from any event a to itself in 0 time, so $\tau(a,a) = 0$.

If an event a casually precedes b, and b causally precedes c, then some trajectories going from a to c pass through b but not necessarily all of them. The largest proper time of trajectories going from a to c via b is composed of the largest time $\tau(a,b)$ of going from a to b and the largest time $\tau(b,c)$ of going from b to c, and is thus equal to $\tau(a,b) + \tau(b,c)$. The largest proper time $\tau(a,c)$ among *all* possible trajectories leading from a to c is larger than or equal to the largest possible time $\tau(a,b) + \tau(b,c)$ among those trajectories which pass through b.

In terms of the largest proper time $\tau(a,b)$, we can conclude that a precedes b if $\tau(a,b) > 0$ — because otherwise, if a does not causally precedes b, we take $\tau(a,b) = 0$. Thus, we conclude that if $\tau(a,b) > 0$ and $\tau(b,c) > 0$, then

$$\tau(a,c) \geq \tau(a,b) + \tau(b,c) \quad (1)$$

Proper time vs. distance. For distance $d(a,b)$, a similar logic based on its definition as the *shortest* path leads to the triangle inequality $d(a,c) \leq d(a,b) + d(b,c)$ — since the length $d(a,c)$ of the shortest path from a to c is smaller than or equal to the shortest length $d(a,b) + d(b,c)$ of all the paths that go through b.

Because of this connection, the above inequality (1) is called an *anti-triangle inequality.* Because of its origin in kinematics (a physical description of how particles and physical bodies move), the function $\tau(a,b)$ is known as a *kinematic (k-)metric*; see, e.g., [1,3,5].

So, we arrive at the following definition.

Definition 1. *Let X be a set. By a* kinematic (k-)metric *of X, we mean a function $\tau : X \times X \to \mathbb{R}$ for which $\tau(a,b) \geq 0$ for all a and b, $\tau(a,a) = 0$, and the following property is satisfied:*

$$\text{if } \tau(a,b) > 0 \text{ and } \tau(b,c) > 0, \text{ then } \tau(a,c) \geq \tau(a,b) + \tau(b,c), \tag{2}$$

The pair (X, τ) is called a k-space-time.

Discussion. The usual metric $d(a,b)$ is symmetric: $d(a,b) = d(b,a)$. In contrast, a k-metric is highly asymmetric: one can easily prove, by contradiction, that if $\tau(a,b) > 0$ then $\tau(b,a) = 0$. Indeed, if we had $\tau(b,a) > 0$, then due to the anti-triangle inequality, we would have

$$0 = \tau(a,a) \geq \tau(a,b) + \tau(b,a) > 0,$$

a contradiction.

A natural question. Since k-space-times are natural models of space-times, a natural question is: is there a *universal* k-space-time, i.e., a k-space-time into which every other k-space-time can be isometrically embedded?

In this paper, we prove that such a universal k-space-time is indeed possible. To formulate our precise result, however, we need to make one physics-related comment.

A physics-related comment. From the mathematical viewpoint, a space-time has *infinitely many* possible events. For example, in Newton's physics or in relativistic physics, we can have events corresponding to infinitely many moments of time and infinitely many spatial points.

In practice, however, at any given moment of time, we have only made *finitely many* observations; in other words, we only have finitely many observed events. This number can be huge, and by performing additional observations and measurements, we can make it even larger — but we will always have finitely many events. Thus, from the operationalist physical viewpoint, it is sufficient to make sure that every *finite* k-space-time can be embedded.

In the metric case, we can make the same restriction and then tend to the limit, since the metric $d(a,b)$ naturally induces the notion of convergence. There is no such automatic convergence for kinematic metric, so we will restrict ourselves to finite k-space-times.

2. Main Result

Definition 2. *Let* (X, τ) *and* (X', τ') *be k-space-times. A 1-1 mapping* $f : X \to X'$ *is called an* isometric embedding *if for every* $x, y \in X$*, we have* $\tau(x, y) = \tau'(f(x), f(y))$*. When* f *is a bijection, we say that the k-metric space-times are* isometric.

Definition 3. *A k-space-time* U *is called* universal *if every finite k-space-time can be isometrically embedded into* U*.*

Definition 4. *Let* (X, τ) *be a k-space-time, and* $S, S' \subset X$*. A bijection* $f : S \to S'$ *is called an* isometry *if for every two points* $x, y \in S$*, we have* $\tau(x, y) = \tau'(f(x), f(y))$*. When* $S = S' = X$*, an isomorphic embedding is called an* auto-isometry.

Definition 5. *A k-space-time* (X, τ) *is called* ultra-homogeneous *if every isometry between its finite subsets* S *and* S' *can be extended to an auto-isometry of* X*.*

Theorem.

- *There exists a universal ultra-homogeneous kinematic metric space.*
- *Every two universal ultra-homogeneous kinematic metric spaces are isometric to each other.*

Comments.

- In other words, modulo isometry, there is a unique universal ultra-homogeneous kinematic metric space.
- In [4, 12], it is shown that a universal ultra-homogeneous space-time model exists if we only model causality relation. In this paper, we extend this universality result to the case when we also consider proper time (i.e., k-metric).

Proof. To prove our result, we will use the Fraïssé limit theory; see, e.g., [7, 8, 11]. In our description, we will only need a relational part of this theory. In this part, we start with a set I which either coincides with the set of all positive integers, or has the form $I = \{1, \ldots, n\}$ for some n. A *signature* L is then defined as a mapping which assigns, to each integer $i \in I$, a symbol R_i and a positive integer n_i called *arity*. By an *L-structure* $\mathbf{A}$, we mean a tuple consisting of a non-empty set A and a sequence of n_i-ary relations $R_i^{\mathbf{A}} \subseteq A^{n_i}$. An injection $f : A \to B$ is called an *embedding* between structures $\mathbf{A}$ and $\mathbf{B}$ if for every i and for all possible values $a_1, \ldots, a_{n_i}$, we have $R_i^{\mathbf{A}}(a_1, \ldots, a_{n_i}) \Leftrightarrow R_i^{\mathbf{B}}(f(a_1), \ldots, f(a_{n_i}))$. The existence of an embedding is denoted by $\mathbf{A} \leq \mathbf{B}$.

An embedding is called an *isomorphism* if it is a bijection. The existence of an isomorphism is denoted by $\mathbf{A} \cong \mathbf{B}$.

A class $\mathcal{K}$ of finite L-structures is called *hereditary* if $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \in \mathcal{K}$ imply $\mathbf{A} \in \mathcal{K}$. It is said to satisfy the *joint embedding property* if for every two structures $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B} \in \mathcal{K}$, there exist a $\mathbf{C} \in \mathcal{K}$ for which $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \leq \mathbf{C}$. The class is said to satisfy the *amalgamation property* if for every three structures $\mathbf{A} \in \mathcal{K}$, $\mathbf{B} \in \mathcal{K}$, and $\mathbf{C} \in \mathcal{K}$, and for every two embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exists a structure $\mathbf{D} \in \mathcal{K}$ and embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ for which $r \circ f = s \circ g$. A countable class $\mathcal{K}$ is called a *Fraïssé class* if it is hereditary, satisfies joint embedding and amalgamation properties, and contains structures with arbitrary large number of elements. Fraïssé has proven that for every Fraïssé class $\mathcal{K}$, there exists a countable ultrahomogeneous structure into which each structure from the class $\mathcal{K}$ can be isomorphically embedded, and that such a structure is unique up to an isomorphism; this universal structure is known as the *Fraïssé limit* of the class $\mathcal{K}$.

We want to use this result to prove our theorem. The main difference between Fraïssé's formulation and our problem is that Fraïssé considers sets with relations, while we are interested in sets with a real-valued function $\tau(a, b)$. So, to reduce our problem to Fraïssé's, we need to describe the real-valued function in terms of relations. We can do it by considering, for each positive rational number $q = \frac{m}{n}$, a binary relation $R_q(a, b) \Leftrightarrow \left(\tau(a, b) \geq \frac{m}{n}\right)$. Since there are countably many rational numbers, we are thus introducing countably many relations.

For every a and b, if we know the truth values of all the relations $R_q(a, b)$, this means that we know the set of all rational numbers $q \leq \tau(a, b)$. From this set, the real number $\tau(a, b)$ can be determined as the supremum of this set. Thus, if $R_q(a, b) \Leftrightarrow R_q(f(a), f(b))$ for all q, this means that the corresponding sets of rational numbers coincide and therefore, $\tau(a, b) = \tau(f(a), f(b))$. So, for the above-defined signature (set of relations), isomorphism means isometry of the corresponding k-space-times, embedding means isometric embedding, etc.

Thus, to apply the Fraïssé result to our case, it sufficient to prove that the class $\mathcal{K}$ of all finite k-space-times is hereditary, has joint embedding and amalgamation properties, and has structures (k-space-times) with arbitrarily large number of elements.

Hereditary property follows directly from the definition of a k-space-time; as k-spaces-times with arbitrarily large number of elements, we can take, e.g., finite subsets of the usual Minkowski space-time — or, even simpler, sets $\{1, \ldots, N\}$ with $\tau(x, y) = \max(y - x, 0)$.

The joint embedding property is also easy to prove: for each pair of structures $\mathbf{A} = (A, \tau_A)$ and $\mathbf{B} = (B, \tau_B)$, as the desired structure $\mathbf{C}$, we can simply take the union C of two disjoint sets A and B, with the following k-metric:

- for $a, a' \in A$, we take $\tau(a, a') = \tau_A(a, a')$;
- for $b, b' \in B$, we take $\tau(b, b') = \tau_B(b, b')$; and

- for $a \in A$ and $b \in B$, we take $\tau(a, b) = 0$.

To complete the proof, let us prove the amalgamation property. Let us assume that two embeddings: $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, for some structures $\mathbf{A} = (A, \tau_A)$, $\mathbf{B} = (B, \tau_B)$, and $\mathbf{C} = (C, \tau_C)$. Without losing generality, we can safely assume that A, B, and C are disjoint sets — otherwise we would simply take disjoint copies of A, B, and C. Then, to get D, we do the following:

- We take the union $B \cup C$.

- On this union, we have an equivalence relation: two points x, x' are equivalent if there exists an $a \in A$ such that $x = f(a)$ and $x' = g(a)$.

One can easily check that this is indeed an equivalent relation, in which each equivalence class consists of either one element or of two elements $\{f(a), g(a)\}$ for some $a \in A$. A factor-set of $B \cup C$ over this equivalence relation is our set D.

The subset of D formed by elements of the type $f(a)$ is isomorphic (thus isometric) to A. Trivial embeddings $r(b) \stackrel{\text{def}}{=} b$ and $s(c) \stackrel{\text{def}}{=} c$ are isomorphic embeddings of B and C into this set D. (In topology, what we do to get D is called “gluing”: we take B and C, and glue them together by the part isomorphic to A.)

The k-metric τ on this set D is then defined as follows:

- for $b, b' \in B$, we take $\tau(b, b') = \tau_B(b, b')$; for $c, c' \in C$, we take $\tau(c, c') = \tau_C(c, c')$;

- for values $b \in B - C$ and $c \in C - B$, we define

$$\tau(b, c) = \max_{a \in A}(\tau_B(b, a) + \tau_C(a, c));$$

- for values $b \in C - B$ and $c \in B - C$, we define

$$\tau(b, c) = \max_{a \in A}(\tau_C(b, a) + \tau_B(a, c)).$$

One can check that thus defined function satisfies the anti-triangle property (2). The statement is proven, and so is our main result.

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