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ON HYPERBOLIC MOTION IN TWO HOMOGENEOUS SPACE TIMES (RESEARCH ANNOUNCEMENT)

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Abstract. In 1960 W. Rindler generalized the concept of hyperbolic motion to an arbitrary Lorentzian manifold and studied this motion in the case of de Sitter space-time. We specify Rindlers (non-linear) system of differential equations in the case of the Segals compact cosmos D (which is locally isometric to the Einstein static universe), and in the case of the Heraclitian space-time F. This F is the real Lie group U(1,1) with a certain bi-invariant metric on it whereas D is U(2) with a bi-invariant metric on it. In each case, we present a particular solution to the Rindlers system.

Keywords: Lie group, hyperbolic motion, Lorentzian manifold.

1. Motivation and Introduction

This work is partly motivated by publication [7] where W. Rindler says (p.2082) that "in the special theory of relativity the term 'hyperbolic motion' is commonly applied to the motion of a test particle moving with constant proper acceleration along a straight line in a suitable Galilean frame of reference. (Proper acceleration is the acceleration relative to the instantaneous Galilean rest frame.) Hyperbolic motion was first noted by Minkowski [4] and was further studied by Born [2], who also coined its name. This name derives from the fact that the plot of distance against time is a rectangular hyperbola (see equation (9) of [7]). By the same terminology the classical motion with constant acceleration is 'parabolic'."

Let us notice that (as it is obvious from [7, p.2083] calculations) a Galilean frame (from above) is another name for an inertial frame in special relativity theory. W. Rindler generalizes the concept of hyperbolic motion to a general space-time ([7, p.2083]) by which (as it becomes clear from [7, p.2084]) Rindler understands a manifold with Lorentzian metric on it. In his article he only solves the proposed equations (that is, our (1.4) below) for the particular case of de Sitter space-time ([7, p.2085]).

Again, the equations mentioned above are deduced as the ones which describe the motion of a uniformly accelerated (point-like) particle. In order to present these equations let us first describe our notation(s). Let a particle (in a portion of space-time with coordinates x^0, x^1, x^2, x^3) have world line

$$x^m = x^m(p) \tag{1.1}$$

where parameter p is the arc length. Velocity U and acceleration A are defined ([7, p.2084]) as follows:

$$U^m = \frac{dx^m}{dp}, \tag{1.2}$$

$$A^m = \frac{DU^m}{dp} \tag{1.3}$$

where the uppercase D is an indication of covariant differentiation. The Rindlers equations ([7, (16) on p.2084]) read as follows:

$$\frac{DA^m}{dp} = a^2 U^m \tag{1.4}$$

where a (positive) constant a is the magnitude of the acceleration A given by (1.3).W. Rindler only analyzed these equations for the de Sitter world (which is a well-known example of a homogeneous space-time).

We introduce below two homogeneous space-times, D and F. Each of them can be viewed as a four-dimensional real Lie group equipped with certain bi-invariant metric of Lorentzian signature. In both cases we specify equations (1.4) and present particular solutions.

2. Space-Times D and F

The Lie groups U(2) and U(1,1) can each be defined as the totality of all 2 by 2 matrices Z (with complex entries allowed) which satisfy

$$Z^* s Z = s \tag{2.1}$$

where s is the unit matrix in case of U(2), and in case of U(1,1) s is the diagonal two by two matrix with entries 1,-1. To make U(2) and U(1,1) space-times, we only need to supply them with metric tensors of Lorentzian signatures: see Section 3.1 (respectively, 3.2) of [3] for these (and more) details on U(2) = D (respectively, U(1,1) = F). At this point in our article, we only need to know that a left-invariant orthonormal basis of vector fields X_0, X_1, X_2, X_3 is chosen on D, and of H_0, H_1, H_2, H_3 on F. In each case, the choice of signature is +, −, −, − . The corresponding metric is bi-invariant (on each of the two groups). The space-time F is known as a tachyonic fluid, [3, Theorem 10] (see more details in that Theorem 10 which justify the $Heraclitian$ world name).

To specify equations (1.4), we will use the following result from [5, Section 8]: in terms of such a basis of vector fields, each Christoffel symbol G^k_{ij} is nothing but one-half of the structure constant C^k_{ij}.

Recall that the structure constants C^k_{ij} are detected from the commutation relations

$$[X_i, X_j] = C^k_{ij} X_k \tag{2.2}$$

(where summation in k goes from k = 0 to k = 3). The Christoffel symbols G^k_{ij} are coefficients in the decomposition of the covariant derivative $D_i(X_j)$ with respect to this very basis of four basic vector fields. Here $D_i(X_j)$ denotes covariant derivative of vector field X_j w.r.t. vector field X_i. Clearly, we have to use vector fields H_i (rather than X_i) when we deal with space-time F. The commutation tables for the two sets of basic vector fields are given in Section 8 of [3].

We can now think of the vector field U from (1.2) as a vector field on the entire space-time, D or F. Each curve of constant acceleration is thus an integral curve of U. By U_0, U_1, U_2, U_3 we now understand coordinates of U with respect to the (above chosen) basis on D (or F). By f′ , f″, etc. below we understand the corresponding (ordinary) derivative of a function f(p) on an integral curve.

Proposition 1. *In the case of D, the vector field U from (1.4) is a solution of the system*
$U_0'' = a^2 U_0,$
$U_1'' + U_2' U_3 - U_2 U_3' = a^2 U_1,$
$U_2'' + U_1 U_3' - U_1' U_3 = a^2 U_2,$
$U_3'' + U_1' U_2 - U_2' U_1 = a^2 U_3.$
The 'F-system' reads as follows:
$U_0'' + U_1' U_2 - U_2' U_1 = a^2 U_0,$
$U_1'' + U_0' U_2 - U_2' U_0 = a^2 U_1,$
$U_2'' + U_1' U_0 - U_0' U_1 = a^2 U_2,$
$U_3'' = a^2 U_3.$

The **proof** is omitted: it is a straightforward application of Milnors result [5, Section 8] to Rindlers system (1.4).

In this article we only discuss the following two solutions. From time to time, we use the abbreviations $C = \cosh(ap), S = \sinh(ap)$.

Proposition 2. *The vector field*

$$U = \{\cosh(ap), \sinh(ap), 0, 0\} = CX_0 + SX_1 \tag{2.3}$$

satisfies the D-system, whereas

$$U = \{\cosh(ap), 0, 0, 0\} = CH_0 \tag{2.4}$$

satisfies the F-system.

The **proof** is an easy verification.

Our next goal is to present the corresponding D-curve which (when $p = 0$) passes through the neutral element of D = U(2). To do so, we use ('excessive') coordinates u_{-1}, u_0, u_1, u_2, u_3, u_4 on D (see [6, p.92]) rather than deal with matrices defined by (2.1).

Theorem 1. *The curve*

$$Z(p) = \left\{ \cos\left[\frac{S}{a}\right], \sin\left[\frac{S}{a}\right], \sin\left[\frac{(C-1)}{a}\right], 0, 0, \cos\left[\frac{(C-1)}{a}\right] \right\} \tag{2.5}$$

is an integral curve of vector field (2.3). It passes (when $p = 0$) through the neutral element of D = U(2).

The **proof** amounts to the direct calculation, and to the careful application of [6, p.92 and p.95] data.

Remark 1. The curve (2.5) is a subset of the 2-dimensional torus T in D: T is defined by equations $u_{-1}^2 + u_0^2 = 1$, $u_1^2 + u_4^2 = 1$, $u_2 = u_3 = 0$.

To deal with the F-system, let us recall the conformal embedding *E* of F into D. This *E* is (indirectly) defined in the proof of [3, Theorem 6]. From that last proof, it follows that an inverse map *G* is defined on the orbit of the four-dimensional group which is generated by vector fields H_0, H_1, H_2, H_3 (which are now viewed as vector fields on D). Here we have the orbit of the U(2) neutral element in mind. A directly defined analogue of *G* is given by formula (3.4) of [1].

Once again, we can use coordinates u_{-1}, u_0, u_1, u_2, u_3, u_4.

Theorem 2. *The image of the curve*

$$Z(p) = \{\cos\left[\frac{S}{a}\right], \sin\left[\frac{S}{a}\right], 0, 0, 0, 1\} \tag{2.6}$$

under G is an integral curve of vector field (2.4).

The **proof** is based on how the vector field $U = CH_0$ is expressed in terms of fields X_0, X_1, X_2, X_3 (see section 3.2 of [3]) and on the conformal embedding *E* of F into D. Once again, it involves application of [6, p.92 and p.95] data and direct calculation.

More details of the Theorem 2 proof are to be presented elsewhere.

Remark 2. It is of interest to consider more details on the F-system within the space-time F itself. To do so, one can start with an explicit formula for the conformal mapping *G* from above.

Remark 3. The curve (2.6) is a geodesic in the space-time D but its image under *G* is not a geodesic in F otherwise it would not be a curve of constant nonzero acceleration in F. In this regard, John Stachel suggested studying how curves of constant acceleration transform when a conformal transformation is applied. Such (and other) questions arise naturally in the scope of his *Unimodular Conformal Projective Relativity* (UCPR), see [8].

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О ГИПЕРБОЛИЧЕСКОМ ДВИЖЕНИИ В ДВУХ ОДНОРОДНЫХ ПРОСТРАНСТВАХ-ВРЕМЕНАХ (АНОНС ИССЛЕДОВАНИЯ)

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Аннотация. В 1960 году В. Риндлер обобщил понятие гиперболического движения для произвольного лоренцева многообразия и изучил это движение в случае пространства-времени де Ситтера. Мы определяем систему (нелинейных) дифференциальных уравнений Риндлера в случае Сигаловского компактного пространства-времени D (которое локально изометрично статической Вселенной Эйнштейна) и в случае Гераклитианского пространства-времени F. При этом пространство-время F является вещественной группой Ли U(1,1) с определённой на ней биинвариантной метрикой, в то время как мир D представляет собой группу Ли U(2) с также определённой на ней биинвариантной метрикой. Для каждого случая представлено частное решение системы Риндлера.

Ключевые слова: группа Ли, гиперболическое движение, лоренцево многообразие.