

CONDITIONAL DIMENSION IN METRIC SPACES: A NATURAL METRIC-SPACE COUNTERPART OF KOLMOGOROV-COMPLEXITY-BASED MUTUAL DIMENSION

V. Kreinovich

Ph.D. (Phys.-Math.), Professor, e-mail: vladik@utep.edu

L. Longpré

Ph.D. (Computer Science), Associate Professor, e-mail: longpre@utep.edu

O. Kosheleva

Ph.D. (Phys.-Math.), Associate Professor, e-mail: olgak@utep.edu

University of Texas at El Paso, El Paso, TX 79968, USA

Abstract. It is known that dimension of a set in a metric space can be characterized in information-related terms – in particular, in terms of Kolmogorov complexity of different points from this set. The notion of Kolmogorov complexity $K(x)$ – the shortest length of a program that generates a sequence x – can be naturally generalized to *conditional* Kolmogorov complexity $K(x : y)$ – the shortest length of a program that generates x by using y as an input. It is therefore reasonable to use conditional Kolmogorov complexity to formulate a conditional analogue of dimension. Such a generalization has indeed been proposed, under the name of *mutual dimension*. However, somewhat surprisingly, this notion was formulated in pure Kolmogorov-complexity terms, without any analysis of possible metric-space meaning. In this paper, we describe the corresponding metric-space notion of conditional dimension – a natural metric-space counterpart of the Kolmogorov-complexity-based mutual dimension.

Keywords: metric dimension, Kolmogorov complexity, conditional Kolmogorov complexity, conditional dimension, mutual dimension.

1. Need for a Metric Analogue of Mutual Dimension: Formulation of a Problem

What is dimension: an informal idea. A straight line segment S_1 is a 1-dimensional set, meaning that to select a point on this segment, it is sufficient to describe the value of a *single* real-valued quantity.

Similarly, a planar area S_2 is a 2-dimensional set meaning that to select a point in this area, we need to describe the values of *two* real-valued quantities: namely, two coordinates of this point.

A spatial area S_3 is a 3-dimensional set meaning that to select a point in this area, we need to describe the values of *three* real-valued quantities: namely, three coordinates of this point, etc.

Metric dimension as a formalization of this informal idea. In practice, we can only describe a real number with some accuracy, and thus, we can only describe a point with some accuracy $\varepsilon > 0$.

Let us start with a straight line segment S_1 of length L . On this segment, two points which are ε -close are, within this accuracy, indistinguishable. So, if we start with a point on this segment, the next ε -distinguishable point has to be at a distance $> \varepsilon$. Thus, the overall number of ε -distinguishable points (or, equivalently, ε -distinguishable real numbers) is

$$N_\varepsilon(S_1) \sim \frac{L}{\varepsilon}.$$

In a 2-D domain S_2 of area A , we can place

$$N_\varepsilon(S_2) \sim \frac{A}{\varepsilon^2}$$

ε -distinguishable points: e.g., we can place such points on a rectangular grid, with $\sim \frac{L}{\varepsilon}$ distinguishable values along each dimension. This number is asymptotically equal to the number $(N_\varepsilon(S_1))^2$ of pairs of ε -distinguishable real numbers – which is in perfect accordance with the fact that we need two real numbers to describe a point in the 2-D domain S_2 .

Similarly, in a 3-D domain S_3 of volume V , we can place

$$N_\varepsilon(S_3) \sim \frac{V}{\varepsilon^3}$$

ε -distinguishable points: e.g., we can place such points on a rectangular grid. This number is asymptotically equal to the number $(N_\varepsilon(S_1))^3$ of triples of ε -distinguishable real numbers – which is also in perfect accordance with the fact that we need three real numbers to describe a point in the 3-D domain S_2 .

For a general metric space, we arrive at the following natural definition.

Definition 1.

- *Let $\varepsilon > 0$ be a real number. We say that a finite set F is an ε -net for the metric space S if every points from S is ε -close to one of the points from the set F .*
- *For each set S in a metric space M , let $N_\varepsilon(S)$ denote the smallest possible number of points in an ε -net of S . We say that the set S has dimension α if $N_\varepsilon(S) \sim \varepsilon^{-\alpha}$ as $\varepsilon \rightarrow 0$.*

Comment. This definition is the main case of the so-called Hausdorff (metric) dimension. This definition goes beyond the usual 1-D, 2-D, and 3-D spaces, since

it is also applicable to irregular sets called *fractals* [3]: e.g., a trajectory of a Brownian motion has dimension $\alpha = 1.5$.

Dimension and information. Dimension can be also described in information terms, namely, in terms of the number of bits (0s or 1s) which are needed to uniquely determine a point in a metric space S with a given accuracy ε .

Specifically, to describe a point with a given accuracy, we need to pinpoint one of the $N_\varepsilon(S)$ ε -close points. If we use b -bit binary strings, then we can identify no more than 2^b different points. Thus, the smallest number of bits b which is needed to identify $N_\varepsilon(S)$ different points is the smallest integer for which $2^b \geq N_\varepsilon(S)$, i.e., the value $b = \lceil \log_2(N_\varepsilon(S)) \rceil$. This logarithm is usually denoted by $H_\varepsilon(S)$ and is called an ε -*entropy* of the metric space.

In terms of the ε -entropy $H_\varepsilon(S) = \log_2(N_\varepsilon(S))$, the definition $N_\varepsilon(S) \sim \varepsilon^{-\alpha}$ takes the form

$$H_\varepsilon(S) \sim -\alpha \cdot \log_2(\varepsilon).$$

Comment. This asymptotic relation is used as an alternative definition of metric dimension.

Relation with Kolmogorov complexity. For subsets of a real line or subsets of an Euclidean space, we can have yet another reformulation of metric dimension: in terms of so-called Kolmogorov complexity $K(x)$. Kolmogorov complexity $K(x)$ of a string x is defined as the shortest length of a program (in some fixed programming language) that is needed to generate the string x ; see, e.g., [2].

To generate all infinitely many bits of a well-defined sequence of bits such as 00... or 0101..., we can use a program of finite length. However, to generate n bits of a truly random sequence x , bits of which do not follow any law, we need to actually list all these n bits in the description of the generating program, so we have $K(x) \geq n$ (to be more precise, $K(x) \geq n - c$ for some constant c). This is exactly why Kolmogorov complexity was invented in the first place: to formally describe the meaning of a random sequence.

In the 1-D case, if we select a random infinite binary sequence

$$x = x_1x_2 \dots x_n \dots$$

that describes a random point in a segment, then for its initial fragments $x_1 \dots x_n$ that describe this point with accuracy 2^{-n} , we get $K(x_1x_2 \dots x_n) \sim n$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{K(x_1 \dots x_n)}{n} = 1.$$

For non-random points, we need fewer bits, so dimension 1 can be defined as the largest possible value of the above limit over all points

$$x = x_1x_2 \dots x_n \dots$$

from this segment.

In the 2-D case, if we select a random point, i.e., a random pair of sequences

$$(x_1, x_2) = (x_{11}x_{12} \dots x_{1n} \dots, x_{21}x_{22} \dots x_{2n} \dots),$$

then, to describe this point with accuracy 2^{-n} , we need $K(x_{11}x_{12} \dots x_{1n}x_{21}x_{22} \dots x_{2n}) \sim 2n$, i.e., we have

$$\lim_{n \rightarrow \infty} \frac{K(x_{11}x_{12} \dots x_{1n}x_{21}x_{22} \dots x_{2n})}{n} = 2.$$

For non-random points, we need fewer bits, so dimension 2 can be defined as the largest possible value of the above limit over all points

$$(x_1, x_2) = (x_{11}x_{12} \dots x_{1n} \dots, x_{21}x_{22} \dots x_{2n} \dots)$$

from the corresponding 2-D domain.

In general, for any d -dimensional point

$$x = (x_1, \dots, x_d) = (x_{11}x_{12} \dots x_{1n} \dots, \dots, x_{d1}x_{d2} \dots x_{dn} \dots)$$

in an Euclidean space, we can define its dimension $\dim(x)$ as the limit

$$\dim(x) = \lim_{n \rightarrow \infty} \frac{K(x_{11}x_{12} \dots x_{1n}x_{21}x_{22} \dots x_{2n} \dots x_{d1}x_{d2} \dots x_{dn})}{n}.$$

Then, for many reasonable sets $S \subseteq \mathbb{R}^d$, the metric dimension is equal to the largest dimension of the corresponding points:

$$\dim(S) = \max_{x \in S} \dim(x).$$

Conditional Kolmogorov complexity and mutual dimension. The notion of Kolmogorov complexity $K(x)$ has been naturally extended to the notion of *conditional* Kolmogorov complexity $K(x : y)$ as the shortest length of a program that, given y as an input, generates x .

In [1], this notion was used to produce the corresponding analog of dimension – which the authors called *mutual dimension*. Specifically, let us consider a Euclidean space $M = \mathbb{R}^d$ which is represented as a Cartesian product $M = X \times Y$, where $X = \mathbb{R}^{d_x}$ and $Y = \mathbb{R}^{d_y}$ with $d_x + d_y = d$. Then, for every point $(x, y) \in M$, we can define *mutual dimension* $\dim(x : y)$ as

$$\dim(x : y) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \frac{K(x_{11} \dots x_{1n}x_{21} \dots x_{d_x 1} \dots x_{d_x n} : y_{11} \dots y_{d_y 1} \dots y_{d_y n})}{n}.$$

Challenge. The problem with the above definition is that it is described in purely Kolmogorov-complexity terms, it is not clear what is its metric analogue of this definition. Such a metric analogue is described in this paper.

2. Conditional Dimension in Metric Spaces: A Natural Definition and its Relation to Kolmogorov-Complexity-Based Mutual Dimension

What is conditional dimension: an intuitive idea. As we have mentioned, the usual metric-space dimension describes how many bits of information we need to describe a point in the given set $S \subseteq M$ with a given accuracy $\varepsilon > 0$.

In this case, a natural interpretation is that points from the set S (or, more generally, from the metric space M) represent physical objects: a point can be an actual point in space, or it can be a point that characterizes a physical object. From the practical viewpoint, this interpretation makes perfect sense:

- all we know about each object is the results of measuring different quantities related to this object,
- so it is natural to represent the object as a tuple consisting of the values of these measurement results.

For example, the state of a point-wise mechanical object can be characterized, at each moment of time, by a tuple consisting of 6 real numbers: 3 spatial coordinates and three components of the velocity vector.

To describe the state of a possibly rotating solid body (e.g., a planet or an asteroid), we need to supplement these 6 numbers with 2 angles describing this body's current orientation and 3 numbers describing its angular velocity.

In many practical situations, we have a system consisting of two interacting subsystems. In such situations, to describe the state of a system, we need to describe the pair (x, y) of states: the state x of the first subsystem and the state y of the second subsystem. In mathematical terms, the set S of all possible states of the system as a whole is thus a subset of the space $M = X \times Y$ of all possible pairs of states, where:

- X is the set of all possible states of the first subsystem and
- Y is the set of all possible states of the second subsystem.

It is natural to ask a question: if we know the state y of the second subsystem, how many bits do we need to describe the state x of the first subsystem?

If the states x and y were unrelated, then of course, the knowledge of y would be of no help. But when they are related, we expect that the knowledge of y can help us find x .

Towards a formal definition. Once we know y , we thus know that the state of possible values of x is limited to the set $\{x : (x, y) \in S\}$. Let us denote this set by S_y . For each y , it is thus reasonable to describe the corresponding number of bits as $\dim(S_y)$.

This number may depend on the choice of y . We want to make sure that the corresponding bound on the number of bits holds for all possible y , so we should

consider the largest of the corresponding values $\dim(S_y)$ as the proper description of the “conditional” dimension.

Thus, we arrive at the following definition.

Definition 2. Let X and Y be metric spaces, and let S be a subset of the set $X \times Y$ of all pairs (x, y) . By the conditional dimension $\dim_{X:Y}(S)$, we mean

$$\dim_{X:Y}(S) \stackrel{\text{def}}{=} \max_{y \in Y} \dim(S_y),$$

where $S_y \stackrel{\text{def}}{=} \{x : (x, y) \in S\}$.

Examples. Let us consider examples in which X and Y are straight line segments and S is a straight line in the rectangle $X \times Y$.

If this straight line is neither parallel to X nor to Y , then the value x is uniquely determined by the value y , i.e., S is a graph of a linear function $x = f(y)$: $S = \{(f(y), y) : y \in Y\}$. In this case, as expected, the conditional dimension is equal to 0: $\dim_{X:Y}(S) = 0 < \dim(\pi_X(S)) = 1$, where $\pi_X(S) = \{x : (x, y) \in S\}$ is the set of all possible x -values, i.e., in mathematical terms, a projection of the set S onto X .

If X and Y are straight line segments and S is a straight line in the rectangle $X \times Y$ which is parallel to Y , then knowing y does not provide us any information about x , so in this case, $\dim_{X:Y}(S) = \dim(\pi_X(S)) = 1$.

If S is the graph of the Brownian motion, i.e., y is time and x is the value of the Brownian motion at time y , then:

- knowing time y , we can uniquely determine the value x , so $\dim_{X:Y}(S) = 0$;
- on the other hand, when we know x , we can only determine y with uncertainty; the corresponding set has dimension 0.5, so $\dim_{Y:X}(S) = 0.5$.

In many of these cases, we have

$$\dim(\pi_X(S)) = \dim_{X:Y}(S) + \dim(\pi_Y(S)),$$

where $\pi_Y(S) \stackrel{\text{def}}{=} \{y : (x, y) \in S\}$ is the projection of the set S on Y . However, sometimes,

$$\dim(\pi_X(S)) < \dim_{X:Y}(S) + \dim(\pi_Y(S)).$$

For example, if S consists of two straight-line segments, one parallel to X and one parallel to Y , then $\dim_{X:Y}(S) = 1$ and $\dim(\pi_Y(S)) = 1$, but

$$\dim(\pi_X(S)) = 1 < \dim_{X:Y}(S) + \dim(\pi_Y(S)) = 1 + 1 = 2.$$

Relation to Kolmogorov complexity. In many of the above examples, we have $\dim_{X:Y}(S) = \max_{(x,y) \in S} \dim(x : y)$.

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УСЛОВНАЯ РАЗМЕРНОСТЬ В МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ: ЕСТЕСТВЕННЫЙ АНАЛОГ ВЗАИМНОЙ РАЗМЕРНОСТИ, ОСНОВАННОЙ НА КОЛМАГОРОВСКОЙ СЛОЖНОСТИ, ДЛЯ МЕТРИЧЕСКОГО ПРОСТРАНСТВА

В. Крейнович

к.ф.-м.н., профессор, e-mail: vladik@utep.edu

Л. Лонгпре

к.т.н., доцент, e-mail: longpre@utep.edu

О. Кошелева

к.ф.-м.н., доцент, e-mail: olgak@utep.edu

Техасский университет в Эль Пасо, США

Аннотация. Известно, что размерность множества в метрическом пространстве можно охарактеризовать в терминах, связанных с информацией — в частности, в терминах колмогоровской сложности различных точек множества. Понятие колмогоровской сложности $K(x)$ — наименьшая длина программы, которая генерирует последовательность x — может быть естественным образом обобщено в условную колмогоровскую сложность $K(x : y)$ — наименьшая длина программы, которая генерирует x , используя y в качестве входных данных. Поэтому целесообразно использовать условную колмогоровскую сложность, чтобы сформулировать условный аналог размерности. Такое обобщение действительно было предложено под названием взаимной размерности. Однако это понятие несколько неожиданно было сформулировано в терминах чистой колмогоровской сложности, без какого-либо анализа возможного значения для метрического пространства. В этой статье мы описываем соответствующее понятие условной размерности для метрического пространства — естественный аналог взаимной размерности, основанной на колмагоровской сложности.

Ключевые слова: метрическая размерность, колмогоровская сложность, условная колмогоровская сложность, условная размерность, взаимная размерность.