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COARSELY GEODESIC METRICS ON REDUCTIVE GROUPS (AFTER H. ABELS AND G. A. MARGULIS)

G.A. Noskov

В статье изучаются свойства функций длины на группах.

We are going to study the length functions on a group G, that is the functions $g \to |g| : G \to \mathbb{R}$, satisfying the following axioms:

- Positivity: $|1| = 0$ and $|g| > 0$ for all nonidentical $g \in G$;
- Triangle inequality: $|gh| \leq |g| + |h|, \quad g, h \in G$;
- Symmetry: $|g| = |g^{-1}|, \quad g \in G$.

Any length function gives rise to a left invariant metric d on G as usual: $d(g, h) \stackrel{def}{=} |g^{-1}h|$. And conversely, any left invariant metric defines a length function.

There are plenty of left invariant Riemannian metrics on any connected real Lie group G, although all of them give rise to a Lipshitz equivalent distance functions. On the other hand if Ω is a compact symmetric neighbourhood of identity $1 \in G$ then we can define a word length function on G by $|g| = \min\{i \in \mathbb{Z}_+ \mid g \in \Omega^i\}$. One can easily see that d is quasi-isometric to the metric induced by any left invariant Riemannian metric on G. Thus the problem of classification of metrics of above type up to a quasi-isometry is trivial - any two of them are quasi-isometric. Both two classes of metrics, introduced above, have a common feature – they are «coarsely geodesic» (see below the definition).

Recently H. Abels and G. A. Margulis gave much more refined classification of coarsely geodesic left invariant proper metrics on reductive Lie groups up to coarse equivalence [1]. They have defined a class of so called normlike pseudometrics on a reductive group, and have proved the following theorem.

Theorem 1. *Let $\mathbb{G}$ be a reductive $\mathbb{R}$-group and $G = \mathbb{G}(\mathbb{R})^0$. Then any left invariant coarsely geodesic proper metric is bounded distance away from a unique normlike pseudometric on G.*

Omsk Branch of Institute of Mathematics, and Mathematisches Institute der Heinrich-Heine-Universitat Duesseldorf.

E-mail: `noskov@private.omsk.su`

This theorem gives the picture of a metric «in a large scale» and favorably compare to the following result of V . Berestovsky which is «local» in nature [2] (for example it does not apply to word metrics):

Theorem 2. *Any interior left invariant metric on a Lie group is the Carnot-Caratheodory-Finsler one.*

We give an exposition of Abels & Margulis result, sacrificing some generality, but not the ideas.

1. Coarse world

Definition 1. A parameterized curve (not necessarily continuous!)

$$\alpha(t) : [0, t_0] \to X$$

in a metric space (X, d) is called a C-coarse geodesic, $C \geq 0$ if $d(\alpha(s), \alpha(t)) \overset{C}{=} |s - t|$ for all $s, t \in [0, t_0]$. (The symbol $\overset{C}{=}$ means equality up to an error not exceeding C.) The space (X, d) is called C-coarsely geodesic, if any two points $x, y \in X$ can be connected by a C-coarse geodesic.

A map $f : X \to Y$ of metric spaces is (A, B)-uniform, $A, B \geq 0$ if

$$\forall\ x, x' \in X : d(x, x') \leq A \Rightarrow d(f(x), f(x')) \leq B$$

and f is called uniform if $\forall\ A \geq 0$ there is $B \geq 0$, such that f is (A, B)-uniform. For example, any C-coarse geodesic is $(B, B + C)$-uniform for any $B \geq 0$.

A map $f : X \to Y$ of metric spaces is proper if the preimage of a bounded set is bounded.

We say that the metrics d_1, d_2 on a space X are Hausdorff or coarsely equivalent iff $|d_1 - d_2|$ is a bounded function on $X \times X$.

2. Reflection groups

We recall notation and facts about finite groups generated by reflections and which we use later on. Sufficient references for this are [3, 4].

Let V be a Euclidean space, i.e. a finite dimensional vector space with an inner product. The reflection in a hyperplane H is the linear transformation $s_H : V \to V$ which is the identity on H and is multiplication by -1 on the (one-dimensional) orthogonal complement $H^{\perp}$. A finite reflection group is a finite group W of linear transformations generated by reflections. We call the fixed-point subspace $V_0 = V^W$ the inessential part of V, and its orthogonal complement V_1 the essential part of V. Let $\mathcal{H}$ denote the set of all reflecting hyperplanes H with in $s_H \in W$. For a reflection $s \in S$, we denote its reflecting hyperplane by H^s. The connected components of the complement of the union $\bigcup\limits_{H \in \mathcal{H}} H$ in $\mathbb{R}^n$ are called Weyl chambers, and the closure of a Weyl chamber is called a closed Weyl chamber. Any closed Weyl chamber A^+ is a fundamental set for W in $\mathbb{R}^n$, i.e. each W-orbit has precisely one point in A^+.

In particular we have a Coxeter projection $\mathfrak{c} : \mathbb{R}^n \to A^+$. It implies that W acts simply transitively on the set of Weyl chambers or, equivalently, any closed Weyl chamber can be uniquely represented as wA^+, $w \in W$. The hyperplane L_s defines two closed halfspaces of $\mathbb{R}^n$ and those one, containing A^+, we denote by L_s^+. We fix a W-invariant inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. For each reflection s there is a unique unit vector α_s, pointing to L_s^+ and orthogonal to L_s and we call this vector an s-root. In terms of roots $A^+ = \{x \in \mathbb{R}^n \mid \langle x, \alpha_s\rangle \geq 0 \quad \forall\, s \in S_0\}$.

Any Weyl chamber is a unique irreducible intersection of halfspaces and we call the corresponding hyperplanes the walls of the chamber. The set S_0 of all reflections in the walls of A^+ generates W. The vectors $\{\alpha_s \mid s \in S_0\}$ form a basis of V_1. Therefore $V_1^\perp \cap A^+$ is a simplicial cone of dimension $|S_0|$, and we have an orthogonal decomposition $A^+ = V_0 + (V_1 \cap A^+)$. We also recall that the angle between any two roots, corresponding to different walls of A^+ is obtuse.

We call convex cone A^{++} spanned by $\{\alpha_s \mid s \in S_0\}$ the dual of A^+. The linear functional ℓ on V is called positive with respect to A^+ if it takes nonnegative values on A^{++}. Let $W_x = \{w \in W \mid wx = x\}$ denote the stabilizer of a point $x \in \mathbb{R}^n$ in W. It is known that W_x is generated by $W_x \cap S_0$ for any $x \in A^+$.

Lemma 1. (Supporting functionals) *Let $\|\cdot\|$ be a W-invariant norm on V. For any nonzero $x \in V$ one can find a linear functional ℓ_x on $\mathbb{R}^n$ such that*

$$\|\ell\| = 1, \quad \ell_x(x) = \|x\|,$$

and ℓ_x is positive with respect to any Weyl chamber containing x.

Proof. It follows from the Hahn-Banach theorem that there exists a linear functional ℓ_x on $\mathbb{R}^n$ such that $\|\ell_x\| = 1$ and $\ell_x(x) = \|x\|$. Averaging ℓ_x over the stabilizer W_x of x in W and using W-invariance of $\|\cdot\|$, we may assume ℓ_x to be invariant under W_x. Fix a Weyl chamber A^+, containing x. It remains to prove that ℓ_x is positive with respect to A^+, that is $\ell_x(\alpha_s) \geq 0$ for every $s \in S_0$, then by invariance $\ell_x(\alpha_s) = 0$. If $sx \neq x$, then

$$\ell_x(sx) \leq \|\ell_x\|\|sx\| = \|x\| = \ell_x(x),$$

i. e. $\ell_x(x) - \ell_x(sx)$ is positive. Moreover, $x - sx$ is a positive multiple of α_s and therefore $\ell_x(x) - \ell_x(sx)$ is a positive multiple of $\ell_x(\alpha_s)$, hence $\ell_x(\alpha_s)$ is positive and thus ℓ_x is positive with respect to A^+. Since ℓ_x is W_x-invariant we conclude that ℓ_x is positive with respect to any $wA^+, w \in W_x$. It is easy to see that any Weyl chamber, containing x is of the form wA^+ for some $w \in W_x$. ■

3. Length functions on $\mathbb{R}^n$ and stable norms

Let $|\cdot|$ be a length function on additive group $\mathbb{R}^n$. If $a \in \mathbb{R}^n$ then by the triangle inequality the sequence $|ma|, m \in \mathbb{N}$ is subbadditive and therefore the following limit

$$\|a\| \stackrel{def}{=} \lim_{m\to\infty} \frac{|ma|}{m}$$

exists for every $a \in \mathbb{R}^n$ and clearly $|a| \geq \|a\|$. We call $\|a\|$ a stable norm of a. It follows from definition that the stable norm is nonnegative, symmetric, homogeneous, satisfies triangle inequality, but it might happen that the stable norm is not positive.

Definition 2. A length function $|\cdot|$ on $\mathbb{R}^n$ is proper, iff both $|\cdot|$ and any Euclidean metric on $\mathbb{R}^n$ have the same system of bounded subsets.

The definition is stronger that that of given in [1], but this does not affect the results we are going to prove.

We need a criterion for a stable norm to be positive. For example it is so if the length function is coarsely geodesic. We wish to strengthen this remark as follows:

Lemma 2. (When the stable norm is a norm) *Suppose $\mathbb{R}^n$ is given with a proper length function $|\cdot|$ which is «C-uniform» for some $C \geq 0$ in a sense that for each $a \in \mathbb{R}^n$ there is a $(1, C)$-uniform curve $\alpha : [0, |a|] \to \mathbb{R}^n$, starting at 0 and ending distance at most C from a. Then the associated stable norm $\|\cdot\|$ is a norm.*

Proof. It follows easily from properness of the metric that condition of lemma holds for standard Euclidean length function $|\cdot|_e$ with some constant, say C'. In particular for each $a \in \mathbb{R}^n, m \in \mathbb{N}$ there is a $(1, C')$-uniform (with respect to $|\cdot|_e$) curve $\alpha : [0, |a|] \to \mathbb{R}^n$, starting at 0 and ending distance at most C' from ma. The sequence of inequalities

$$m|a|_e = |ma|_e \overset{C'}{=} |\alpha(|ma|)|_e \leq \sum_{i=1}^{[|ma|]} |\alpha(i) - \alpha(i-1)|_e + C' \leq C'([|ma|] + 1)$$

implies that if a is nonzero, then $|ma|$ grows at least linearly with m for nonzero a, hence $\|a\|$ is positive. ■

4. Groups with Cartan projection and stable norms

Definition 3. Suppose that we are given:

1) A group G with a coarsely geodesic length function $|\cdot|$;

2) Its subgroup A with a fixed isomorphism $A \simeq \mathbb{R}^n$;

3) Action of a reflection group W on A and a Weyl chamber A^+.

We say that the map $a(g) : G \to A^+$ is a Cartan projection if the following conditions are satisfied:

CP1) The restriction of the map $a(g)$ to A is a Coxeter projection;

CP2) The map $a(g)$ coarsely preserves the norm $|\cdot|$;

CP3) «Triangle inequality»: $a(g) + a(h) - a(gh) \in A^{++}$ for any $g, h \in G$.

We will call an assembly $\mathcal{G} = (G, |\cdot|, A, W, A^+, a(g))$ a group with Cartan projection. We fix such one throughout this section. Equally with the above notation, we write sometimes, $a(g) \overset{\text{def}}{=} a_g$ for the Cartan projection. Clearly, for any $w \in W$ the map $wa(g) : G \to wA^+$ is again a Cartan projection.

Lemma 3. *If the length function $|\cdot|$ is proper on A then the Cartan projection is uniform.*

Proof. Note that

$$a_{gh} - a_g \in a_h - A^{++}$$

and

$$a_{gh} - a_g \in -a_{h^{-1}} + A^{++}$$

(by Axiom CP3). By properness a_h and $a_{h^{-1}}$ are in a bounded subset of $\mathbb{R}^n$, hence so is the intersection of the right hand sides, since

$$(a - A^{++}) \cap (b + A^{++})$$

is compact for any $a, b \in \mathbb{R}^n$ and hence so is the intersection of the right hand sides. It follows that $|a_{gh} - a_g|$ is bounded whenever $|h|$ is bounded, that is the Cartan projection is uniform. ■

By construction of Section 3. associated to the length function on G is the stable norm on $A \simeq \mathbb{R}^n$ and we are interested when it is positive.

Lemma 4. *The stable norm is positive on A.*

Proof. Fix nonzero $a \in A$ and let $g(t)$, $0 \le t \le |ma|$ be a coarse geodesic in G from 0 to ma. Note that any coarse geodesic is $(1, C)$-uniform for some $C > 0$. Since Cartan projection is uniform, it follows that the projected curve $a_{g(t)}$ is uniform again. It follows that $\|a\|$ is positive by Lemma 2. ■

Lemma 5. *The stable norm on A is W-invariant.*

Proof. Since the restriction of Cartan projection to A is a Coxeter projection, and it coarsely preserves the norm, we see that the restriction of length function to A is coarsely W-invariant. Hence the stable norm is W-invariant. ■

Definition 4. We extend a stable norm onto the whole G via Cartan projection: $\|g\| \stackrel{\text{def}}{=} a_g, g \in G$. Similarly, if ℓ is a functional on $\mathbb{R}^n$, we extend it onto G by precomposing with Cartan projection.

Lemma 6. *Stable norm on G satisfies the triangle inequality:*

$$\|gh\| \le \|g\| + \|h\|$$

for any $g, h \in G$. Moreover for any functional ℓ on $\mathbb{R}^n$, positive with respect to A^+, we have

$$\ell(gh) \le \ell(g) + \ell(h)$$

for any $g, h \in G$.

Proof. Triangle inequality for ℓ immediately follows by application of ℓ to the inclusion in the axiom CP3 above. Let $g, h \in G$. If $\|a_{gh}\|$ is zero then the assertion is obvious, so we assume that it is nonzero. According to Lemma 1, there exists a linear functional ℓ on $\mathbb{R}^n$ such that $\|\ell\| = 1$, $\ell(a_{gh}) = \|a_{gh}\|$, and ℓ is positive with respect to A^+. Then applying triangle inequality for ℓ we obtain

$$\|gh\| \stackrel{\text{def}}{=} \|a_{gh}\| = \ell(gh) \le \ell(g) + \ell(h) \stackrel{\text{def}}{=} \ell(a_g) + \ell(a_h) \le \|a_g\| + \|a_h\| \stackrel{\text{def}}{=} \|g\| + \|h\|.$$

■

5. Miraculous elements

Let $\mathcal{G} = (G, |\cdot|, A, W, A^+, a(g))$ be a group with Cartan projection and let C_0 be a constant, such that the length function is C_0-coarsely geodesic and the Cartan projection C_0-coarsely preserves the length function.

Definition 5. We say that $a \in \mathbb{R}^n$ is C-good, $C \geq 0$, if $\|a\| \geq |a| - C$. We say that a linear functional ℓ is almost supporting at a if $||l|| = 1$ and $\ell_i(a) \geq ||a|| - 1$.

Our aim is to produce $C-$good elements g in «every direction».

Lemma 7. *There is a constant C depending only on $\mathcal{G}$ and on natural n_0 such that for any $a \in A^+$ with $\|a\| > 1$ and for any finite family $\mathcal{L}$ of linear functionals on $\mathbb{R}^n$ coarsely supporting at a, positive with respect to A^+ and of cardinality at most n_0 there is an element $g \in G$ such that,*

$$|a_g| \overset{C}{=} ||a|| \text{ and } \ \ell(a_g) \geq ||a|| - C, \ell \in \mathcal{L}. \tag{1}$$

Proof. Let $g(t) : [0, t_0] \to G$ be a C_0-coarse geodesic in G from 0 to ma, in particular $t_0 \overset{C_0}{=} |ma|$. We define

$$t_j = \frac{t_0}{m} j$$

for $j = 0, \ldots, m$ and we define

$$g_j \overset{\text{def}}{=} g(t_{j-1})^{-1} g(t_j),$$

for $1 \leq j \leq m$. We have $ma = g_1 \cdots g_m$ and

$$|g_j| \overset{\text{def}}{=} d(g(t_{j-1}), g(t_j)) \quad \overset{C_0}{=} \quad \frac{t_0}{m} \quad \overset{C_0/m}{=} \quad \frac{|ma|}{m}$$

for all j. In particular

$$|a_{g_j}| \overset{C_0}{=} |g_j| \overset{C_0}{=} \frac{|ma|}{m} \overset{1}{=} \|a\|$$

for $m \gg 0$ and we conclude that

$$|a_g| \overset{2C_0+1}{=} ||a||$$

for all j and $m \gg 0$ (take into account that the composition of relations $\overset{C}{=}, \overset{D}{=}$ is $\overset{C+D}{=}$). Thus, any g_j for $m \gg 0$ fulfils the first condition in the Lemma. Further, for $\ell \in \mathcal{L}$

$$\ell(g_j) \leq \|a_{g_j}\| \leq |a_{g_j}| \leq |g_j| + C_0 \leq \|a\| + C_0 + 1 \tag{2}$$

for $m \gg 0$. Since any ℓ is coarsely supporting and by triangle inequality from Lemma 6 we have

$$m(\|a\| - 1) \leq \ell(ma) \leq \sum_{j=1}^{m} \ell(g_j). \tag{3}$$

We wish to derive from 2 inequalities above that

$$\ell(g_j) \geq \|a\| - C$$

for all $\ell \in \mathcal{L}$ and universal constant C. For any C and any $\ell \in \mathcal{L}$ we easily conclude from (2),(3) that

$$b_{\ell,m} \stackrel{\text{def}}{=} \#\{j \mid 1 \leq j \leq m,\ \ell(g_j) \leq \|a\| - C\} \leq \frac{m(C_0+2)}{C_0+C+1}.$$

Thus, taking $C \geq (n_0+1)(C_0+2)$ we obtain that

$$\frac{b_{\ell,m}}{m} < \frac{1}{n_0+1}$$

for any $\ell \in \mathcal{L}$ and for sufficiently large m. From this and the definition of $b_{\ell,m}$ we deduce the existence of j, $1 \leq j \leq m$, such that for all $\ell \in \mathcal{L}$, $\ell(g_j) > \|a\| - C$ and we take $g = g_j$. ■

6. Cartan projections of geodesic curves

Let $\mathcal{G} = (G, |\cdot|, A, W, A^+, a(g))$ be a group with Cartan projection and let C_0 be a constant, such that the length function is C_0-coarsely geodesic and the Cartan projection C_0-coarsely preserves the length function. Our aim is to construct a continuous curve, consisting entirely of C-good elements and along which the given finite family of functionals grow coarsely with time.

Lemma 8. *There exists a constant $C > 0$, depending only on $\mathcal{G}$ and on a natural number n_0 such that for any $a \in A^+$ and an arbitrary family $\mathcal{L}$ of coarsely supporting at a functionals, positive with respect to A^+, and of cardinality at most n_0 there exists a continuous curve $\alpha : [0, t_0] \to A^+, t_0 \stackrel{C}{=} |a|$ starting at 0, such that*

$$|\alpha(t)| \stackrel{C}{=} t \quad \textit{and} \quad \ell(\alpha(t)) \geq t - C,\ t \in [0, t_0],\quad \ell \in \mathcal{L}. \tag{4}$$

Proof. Let D be a constant exceeding both C_0 and the constant given by Lemma 7. In particular, there is an element $g \in G$ such that

$$|a_g| \stackrel{D}{=} ||a|| \text{ and } \ \ell(a_g) \geq ||a|| - D,\quad \ell \in \mathcal{L}. \tag{5}$$

Let $g(t) : [0, t_0] \to G$, be a D-coarse geodesic in G connecting 1 and g, in particular $t_0 \stackrel{D}{=} |g|$. Since the Cartan projection coarsely preserves the length and $g(t)$ is a D-coarsely geodesic we conclude that $|a_{g(t)}| \stackrel{D}{=} |g(t)| \stackrel{D}{=} t$ and thus $|a_{g(t)}| \stackrel{2D}{=} t$, hence the curve $a_{g(t)}$ satisfies the first assertion of the Lemma with $C = 2D$.

Let $g'(t) = g(t)^{-1}g$, so that $g = g(t)g'(t)$. For $\ell \in \mathcal{L}$ we have

$$||a|| - D \leq \ell(g) \leq \ell(g(t)) + \ell(g'(t)) \leq \ell(g(t)) + |a_{g'(t)}| \stackrel{D}{=} \ell(g(t)) + |g'(t)| =$$

$$= \ell(g(t)) + d(g(t), g(t_0)) \stackrel{D}{=} \ell(g(t)) + t_0 - t.$$

The equalities $|g| \stackrel{D}{=} ||a||$ and $t_0 \stackrel{D}{=} |g|$ imply $t_0 \stackrel{2D}{=} ||a||$, and substituting this to the above we get

$$\ell(g(t)) \geq t - 5D, \quad 0 \leq t \leq t_0,$$

thus the curve $a_{g(t)}$ satisfies the second assertion with $C = 5D$.

To make $a_{g(t)}$ continuous we define the curve $\alpha(t)$ as the curve which coincides with $a_{g(t)}$ at the integral moments $j, 0 \leq j \leq [t_0]$ and at the moment t_0 and which is linear on the segments $[j, j+1], 0 \leq j \leq [t_0] - 1, \quad [[t_0], t_0]$. First note $g(t)$ and $a(g)$ are uniform maps, so there is a constant E depending only on $\mathcal{G}$, such that that $|\alpha(t) - \alpha([t])| \leq E, t \in [0, t_0]$. Let $C = E + 5D + 1$. We have

$$||\alpha(t)| - t| \leq |\alpha(t) - \alpha([t])| + ||\alpha([t])| - t| \leq E + 2D,$$

hence $|\alpha(t)| \stackrel{C}{=} t$ for all $t \in [0, t_0]$. (We ask the reader to forgive the conflict of notations for length function on G and for absolute value for reals). Finally, for any $t \in [0, t_0]$ and any $\ell \in \mathcal{L}$ we have

$$\ell(\alpha(t)) = \ell(\alpha([t])) + \ell(\alpha(t) - \alpha([t])) \geq [t] - 5D - |\alpha(t) - \alpha([t])| \geq t - E - 5D - 1.$$

■

7. Main Theorem

Theorem 3. *Let $\mathcal{G}$ be a group with Cartan projection. Suppose that the length function is proper on G. Then*

$$\sup_{g \in G} ||g| - \|g\|| < \infty.$$

In particular any left invariant, coarsely geodesic, proper metric on G is bounded distance away from a unique normlike pseudometric on G.

Proof. The main case is $g \in A \simeq \mathbb{R}^n$, from which the general case follows easily by definition of the stable norm on G. Since our length function is proper, $\|\cdot\|$ is a norm. Fix a real $r > 0$ and let $B_r \subset A$ be a unit ball of radius r about origin. In this proof we define a continuous map $\varphi : B_r \to \mathbb{R}^n$, such that $\varphi(B_r)$ consists entirely of C-good points and the image contains $B_{0,r-C}$. The last claim is proved using a topological argument, namely the degree of maps between spheres.

Applying standard arguments, one can show that there exists a triangulation $\mathcal{T}$ of the boundary ∂B_r such that each simplex of $\mathcal{T}$ has $\|\cdot\|$-diameter $\leq 1/2$ and lies entirely in some Weyl chamber. Let $\mathcal{T}_b$ denote the barycentric subdivision of $\mathcal{T}$. For any simplex σ in either $\mathcal{T}$ or $\mathcal{T}_b$, we denote by V_σ the set of vertices in σ. Let σ_x denote the smallest simplex in $\mathcal{T}$ which contains a given $x \in \partial B_r$.

According to Lemma 1, for any nonzero $x \in \mathbb{R}^n$, one can find a linear functional ℓ_x on $\mathbb{R}^n$ such that $\|\ell_x\| = 1$, $\ell_x(x) = \|x\|$, and ℓ_x is positive with respect to any

Weyl chamber containing x. Note that ℓ_x is almost supporting at any point of σ_x. Indeed, for $y \in \sigma_x$ we have

$$\ell_x(y) = \|x\| + \ell_x(y - x) \geq ||y|| - 2||y - x|| \geq \|y\| - 1.$$

For each vertex $v \in \mathcal{T}_b$ we define the set of functionals

$$\mathcal{L}_v = \{\ell_u : u \in V_{\sigma_v}\}.$$

Note that this set satisfies the conditions of the Lemma 8. Indeed, the functionals are of norm 1, they are almost supporting at v by previous remark, and if wA^+ is any Weyl chamber, containing v, then the simplex σ_v is contained in wA^+ too, hence all functionals are positive with respect to wA^+. Now by Lemma 8 for some $C \geq 0$, depending only on $\mathcal{G}$ we find a continuous curve $\alpha : [0, t_v] \to \mathbb{R}^n$, such that

$$|\alpha_v(t)| \stackrel{C}{=} t, \quad 0 \leq t \leq t_v, \quad t_v \stackrel{C}{=} ||v||, \quad \alpha(0) = 0,$$

and

$$\ell_u(\alpha_v(t)) \geq t - C \;\text{ for any } u \in V_{\sigma_v}.$$

We now define a continuous map $\varphi : B_r \to \mathbb{R}^n$ as follows. Take any $z \in \partial B_r$, and let σ be a simplex in $\mathcal{T}_b$ which contains z. Represent z as a convex linear combination

$$z = \sum_{v \in V(\sigma)} \lambda_{v,z} v,$$

and define

$$\varphi(sz) \stackrel{\text{def}}{=} \sum_{v \in V_\sigma} \alpha_v(s\lambda_{v,z} t_v), \quad 0 \leq s \leq 1.$$

It is clear that the nonzero coefficients $\lambda_{v,z}$ in the decomposition above do not depend on the choice of σ. Therefore $\varphi(tz)$ does not depend on the choice of σ either. Since the curves $\{\alpha_v(s)\}$ are continuous, the map $\varphi : B_r \to \mathbb{R}^n$ is continuous.

Claim. *$\varphi(B_r)$ consists of C-good points.*

Since $\mathcal{T}_b$ is the barycentric subdivision of $\mathcal{T}$, one can find a vertex u in σ such that $u \in V_{\sigma_v}$ for all $v \in V_\sigma$. Then it follows

$$||\varphi(sz)|| \geq \ell_u(\varphi(sz)) = \sum_{v \in V(\sigma)} \ell_u(\alpha_v(s\lambda_{v,z} a_v)) \geq \sum_{v \in V_\sigma} s\lambda_{v,z} a_v - C \geq$$

$$\sum_{v \in V_\sigma} |\alpha_v(s\lambda_{v,z} a_v)| - C \geq |\sum_{v \in V_\sigma} \alpha_v(s\lambda_{v,z} a_v)| - C = |\varphi(sz)| - C.$$

Let $\{\varphi_\lambda\}, \lambda \in [0, 1]$ be a linear homotopy between identity and ϕ. It follows from inequalities above that

$$\|\varphi_\lambda(z)\| \geq \ell_u(\varphi_\lambda(z)) = \ell_u(\lambda\varphi(z) + (1 - \lambda)z) > r - C$$

for any $z \in \partial B_r$ and $0 \leq \lambda \leq 1$. We get from above that

$$B_{r-C-1} \cap \varphi_\lambda(\partial B_r)$$

for any $0 \leq \lambda \leq 1$. The following topological lemma finish the proof.

Lemma 9. *Suppose that a continuous map $f : B_r \to \mathbb{R}^n$ satisfies the following property: t $B_{r-C-1} \cap f_\lambda(\partial B_r))$ for any $0 \leq \lambda \leq 1$, where f_λ is a linear homotopy between identity and f. Then the image $f(B_r)$ contains B_{r-C-1} for all r sufficiently large.*

Proof. Arguing by contradiction, suppose that $f(B_r)$ does not contain a point $y \in B_{r-C-1}$. By assumption the restrictions of Id and f to ∂B_r are homotopic as maps into $\mathbb{R}^n - \{y\}$. Composing with projection map of the last space onto the sphere S_r, we obtain that the identity map of the sphere is homotopic to the constant map - this contradicts to the well known fact (the continuous maps of the sphere into itself of different degree are not homotopic). This contradiction proves the Lemma. ∎

Theorem is proved. ∎

8. Examples of groups with Cartan projection

The notion of a group with a Cartan projection is motivated by the classical Cartan decomposition for semisimple or, more generally, reductive Lie groups. Let

$$G = \mathbb{G}(\mathbb{R})^0$$

be the connected component of the identity of the group $\mathbb{G}(\mathbb{R})$ of $\mathbb{R}$-rational points of a reductive $\mathbb{R}$-group $\mathbb{G}$. Fix a proper coarsely geodesic length function $|\cdot|$ on G. Let $\mathbb{A}$ be a maximal $\mathbb{R}$-split torus in $\mathbb{G}$ and $A = \mathbb{A}(\mathbb{R})^0$. The group A is isomorphic to $\mathbb{R}^n$ where $n = \dim A$. Let $A^+ \subset A$ be a Weyl chamber for the Weyl group $W = \mathcal{N}_G(A)/\mathcal{Z}_G(A)$.

Theorem 4. *The assembly $\mathcal{G} = (G, |\cdot|, A, W, A^+, a(g))$ is a group with Cartan projection.*

Proof. It is well known that G admits a Cartan decomposition $G = KA^+K$, where K is a suitable maximal compact subgroup of G and that associated «Cartan projection»

$$a(g) : G \to A^+$$

is well defined. Moreover, if $w \in W$ and $a \in A^+$ then $w(a) \in KaK$, and this implies that the restriction of $a(g)$ onto A is a Coxeter projection, thus CP1) is fulfilled. Since K is compact and the length function is proper, it follows that the Cartan projection coarsely preserves the length function, hence CP2).

The axiom CP3) is rather nontrivial and relies on the presentation of positive linear functionals as linear combinations of highest weights of rational representations [5]. Let $\pi : G \to GL(V)$ be a rational representation of the group G defined and irreducible over $\mathbb{R}$. We decompose V into the direct sum

$$V = \bigoplus_{\chi \in X(A)} V_\chi, \quad V_\chi \neq \{0\}, \tag{6}$$

of the weight spaces

$$V_\chi \stackrel{\text{def}}{=} \{v \in V \mid \pi(a)v = \chi(a)v \quad \text{for any} \quad a \in A\},$$

where $X(A)$ is the group of rational characters of A. Let $\mu_\pi \in X(A)$ be the highest weight of the representation π. Then

$$\chi(a) \leq \mu_\pi(a) \quad \text{if} \quad a \in A^+ \quad \text{and} \quad V_\chi \neq \{0\}. \tag{7}$$

It is well known that one can introduce a $\pi(K)$-invariant inner product on V with respect to which the transformations $\pi(a), a \in A$, are self-adjoint. Then the subspaces V_χ in the decomposition (6) are mutually orthogonal, and it follows from (7) that $\|\pi(g)\| = \mu_\pi(a(g))$, for any $g \in G$, where the norm is taken with respect to the inner product just defined. As a consequence, we get that for any $g, h \in G$,

$$\log \mu_\pi(a(gh)) \leq \log \mu_\pi(a(g)) + \log \mu_\pi(a(h)). \tag{8}$$

It is well known that any linear functional ℓ on $A = \mathbb{R}^n$, which is positive with respect to A^+, can be represented as a positive linear combination

$$\ell = \sum_{i=1}^{m} b_i \log \mu_{\pi_i}, \; b_i \geq 0,$$

where $\pi_i, 1 \leq i \leq m$, are rational representations of G, defined and irreducible over $\mathbb{R}$. Then it follows from (8) that for any $g, h \in G$,

$$\ell(a(gh)) \leq \ell(a(g)) + \ell(a(h)). \tag{9}$$

Since ℓ is an arbitrary positive functional we get from this and the definition of A^{++} that the axiom CP3) is satisfied. ■

9. Generalizations, questions, problems

The notion of a group with a Cartan projection used above is not enough to treat the case of reductive groups over local fields. The generalization is given in [1] and here we give a sketch. Namely we must allow not only $\mathbb{R}^n$ but any closed cocompact group D of $\mathbb{R}^n$. An action of a reflection group W on $\mathbb{R}^n$ should leave D invariant and there must be a compact symmetric set $M \subseteq \mathbb{R}^n$, containing 0, such that for each $w \in W$ there is an inclusion

$$wA^+ \subseteq (D \cap wA^+) + M.$$

We say that the map

$$a(g) : G \to A^+$$

is a Cartan projection if CP1), CP2) are satisfied and CP3) is satisfied in the following relaxed form:

$$a(g) + a(h) - a(gh) \in A^{++} + M$$

for any $g, h \in G$. The generalization includes the case of invariant metrics on $\mathbb{Z}^n$ as well as invariant metrics on $\mathbb{R}^n$, but seems it does not include the case of $\mathbb{Z}^n$-invariant metrics on $\mathbb{R}^n$, which were treated by D. Burago [6].

Question 1. Is the analog of Abels-Margulis theorem for lattices in a reductive group valid? (Presumably not). The same question for the metrics on a reductive group invariant under translations by the elements of the lattice. (The answer is presumably yes).

Question 2. What is the relation of the triangle inequality for stable norm to the Gelfand-Naimark theorem about singular values of the product of matrices?

Question 3. What is the analog of Abels-Margulis theorem for nilpotent, solvable, general Lie groups? Look at the S. Krat paper.

Question 4 What is the structure of the asymptotic cone of a reductive group with a normlike pseudometric? Presumably they are Minkowski buildings. The question is related to the results of Kleiner-Leeb, Thornton, Parreaut.

Question 5 What is the relation between Abels-Margulis and Berestovsky theorems? For example, could one calculate a normlike pseudometric associated to a Carnot-Caratheodory-Finslerian metric on a reductive group?

Question 6. Let M be a set provided with two interior metrics d_1 and d_2. Assume that a group G acts cocompactly on M by isometries with respect to both metrics and

$$\lim_{d_2(x,y)\to\infty} \frac{d_1(x,y)}{d_2(x,y)} = 1. \tag{10}$$

Due to a result of D. Burago [6], if $G = \mathbb{Z}^n$, then d_1, d_2 are coarsely equivalent. This fact means that all metrics on M diverge linearly or stay within a finite distance from each other. Burago raised the question for which groups the same statement could be true. The Abels-Margulis result easily implies the positive answer for metrics on reductive groups. D. Burago suggested two different directions. The first is the case of semi-hyperbolic groups, i.e., groups of isometries of a space whose curvature is bounded from above by 0. The other one is the case of nilpotent groups and first of all, the Heisenberg group. In some cases the fact that two metrics cannot diverge more slowly than linearly could be described as the finiteness of the Gromov-Hausdorff distance between the group with induced metric and its asymptotic cone. In the case of the abelian group $\mathbb{Z}^n$ the asymptotic cone is $\mathbb{R}^n$ and it lies within a finite Gromov-Hausdorff distance from $\mathbb{Z}^n$. The Gromov-Hausdorff distance between Heisenberg groups and its asymptotic cone is finite [7].

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