

IS LANGRANGIAN FORMALISM ADEQUATELY DESCRIBING ENERGY CONSERVATION?

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In most physical theories, total energy is conserved. For example, when the kinetic energy of a particle decreases, the potential energy increases accordingly. For some physical systems, energy is not conserved. For example, if we consider a particle moving with friction, the energy of the particle itself is not conserved: it is transformed into thermal energy of the surrounding medium. For simple systems, energy is easy to define. For more complex physical systems, such a definition is not easy. To describe energy of generic systems, physicists came up with a general notion of energy based on the Lagrangian formalism — a minimal-action representation of physical theories which is now ubiquitous. For many physical theories, this notion leads to physically meaningful definitions of energy. In this paper, we show that there are also examples when the Lagrangian-motivated notion of energy is not physically meaningful at all — e.g., according to this definition, all dynamical systems are energy-conserving.

1. Energy Conservation: Physical Meaning and Lagrangian-Based Description

Energy conservation: physical meaning. Some physical systems are *conservative* in the sense that their total energy is preserved. For example, the dynamics of a particle in a potential field $V(x) = V(x_1, x_2, x_3)$ is described, in Newtonian mechanics, by Newton's equations

$$m \cdot \ddot{x}_i = -\frac{\partial V}{\partial x_i}, \quad (1)$$

where \dot{x}_i , as usual, denotes time derivative. For this particle, the overall energy

$$E = \frac{1}{2} \cdot m \cdot \sum_{i=1}^3 (\dot{x}_i)^2 + V(x) \quad (2)$$

is conserved: when the kinetic energy $\frac{1}{2} \cdot m \cdot \sum_{i=1}^3 (\dot{x}_i)^2$ decreases, the potential energy $V(x)$ increases appropriately, and vice versa.

When energy is not conserved: physical meaning. A classical example of a physical system for which energy is not conserved is a system with friction. Its simplest case is when we do not even have any potential field, i.e., when the dynamical equations have the form

$$m \cdot \ddot{x}_i = -k \cdot \dot{x}_i, \quad (3)$$

for some friction coefficient k . This equation can be further simplified into

$$\ddot{x}_i = -k_0 \cdot \dot{x}_i, \quad (3a)$$

for $k_0 \stackrel{\text{def}}{=} \frac{k}{m}$. A system that follows this equation slows down, its velocity (and hence, its kinetic energy) exponentially decreases with time — without being transferred into any other type of energy.

From the physical viewpoint, this non-conservation of energy means that the system described by the equation (3a) is not closed: the energy lost in this system is captured by other objects. For friction, it is very clear where this energy goes: it gets transformed into the thermal energy, i.e., into kinetic energy of individual molecules in the surrounding medium.

Need to go beyond simple examples. For simple particles, energy is easy to define and easy to analyze. However, for more complex physical systems, especially when fields are involved, it is not easy to find an appropriate expression for energy.

A general Lagrangian approach to energy conservation. Newton's physics was originally formulated in terms of differential equations. It turns out that most physical theories can be equivalently described in terms of the *minimal action* principle: the actual dynamics of particles and fields is the one that minimizes a special physical quantity called *action* S . For particles, action has the form $S = \int L(x(t), \dot{x}(t)) dt$, where the function $L(x(t), \dot{x}(t))$ is known as the *Lagrangian*. For example, for the Newtonian particle in a potential field $V(x)$, the Lagrangian has the form

$$L = \frac{1}{2} \cdot m \cdot \sum_{i=1}^3 (\dot{x}_i)^2 - V(x). \quad (4)$$

For fields $f(x), \dots$, the action S has a similar form $S = \int L(f(x), \dots, f_{,i}(x), \dots) dx$, where $f_{,i}$ denotes the corresponding partial derivative $f_{,i} \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}$.

The Lagrange formulation of physical theories is currently ubiquitous. One of the main reasons for this ubiquity is that, according to modern physics, the correct picture of the physical world comes from quantum mechanics. It is not easy to find a quantum analogue of a physical theory based on its system of differential equations, but when a physical theory is given in Lagrangian terms, its quantization is much more straightforward: in the Feynman's integration-over-trajectories formulation, the amplitude $\psi_{A,B}$ of a transition from a state A to the state B is proportional to the "sum" (integral) of the expression $\exp\left(i \cdot \frac{S}{\hbar}\right)$ over all trajectories leading from A to B , and the probability to observe the transition into different states B is proportional to the squared absolute value of this amplitude $|\psi_{A,B}|^2$; see, e.g., [2, 2].

Once we know the Lagrangian, we can use Euler-Lagrange equations to derive the corresponding differential equations. For particles, these equations take the form

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = 0. \quad (5)$$

One can easily check that for the Newtonian Lagrangian (4), we get exactly Newton's equations (1). For fields, the equations take the form

$$\frac{\partial L}{\partial f} - \sum_i \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial f_{,i}} \right) = 0. \quad (6)$$

In the Lagrange approach, the energy of a particle is formally defined as

$$E_L \stackrel{\text{def}}{=} \sum_i \dot{x}_i \cdot \frac{\partial L}{\partial \dot{x}_i} - L. \quad (7)$$

One can check that for the Newtonian Lagrangian (4), we get the standard expression (2) for energy. A similar expression defines energy for a field theory [2, 2]:

$$E_L = \sum_f \sum_i f_{,i} \cdot \frac{\partial L}{\partial f_{,i}} - L. \quad (8)$$

What we do in this paper. The Lagrangian approach has been very successful in describing physical energy of different particle and field systems. What we show, however, is that in some simple cases, the Lagrangian formalism does not adequately convey the physical meaning of energy conservation.

2. A Simple Example When the Physical Meaning of Energy Conservation Differs from the Lagrangian-Based Energy

Description of the simple example. Let us consider the simplest possible example of a physical system in which, from the physical viewpoint, energy is

not conserved: a 1-D particle with friction, whose dynamics is described by the equation

$$\ddot{x}(t) = -k_0 \cdot \dot{x}(t). \quad (9)$$

What we will do. In this section, we will show that this system can be described by a Lagrangian and thus, for this system, energy (as defined in the Lagrangian formalism) is well conserved. This will show that — at least on this example — the Lagrangian formalism does not adequately convey the physical meaning of energy conservation.

In the next section, we show that this inadequacy is not a freaky property of this particular simple system: generic dynamical systems describing a 1-D particle can be described by an appropriate Lagrangian.

Towards finding an appropriate Lagrangian. The classical Newtonian Lagrangian (4) is a sum of two terms: a term depending only on \dot{x}_i and a term depending only on x_i . Let us look for a similar type Lagrangian for describing the equation (9), i.e., let us look for a Lagrangian of the type

$$L = a(\dot{x}) + b(x), \quad (10)$$

for some functions $a(\dot{x})$ and $b(x)$. For this Lagrangian, Euler-Lagrange equations (5) lead to

$$b'(x) - \frac{d}{dt}a'(\dot{x}) = 0, \quad (11)$$

where $b'(x)$ and $a'(\dot{x})$, as usual, indicated derivatives of the corresponding functions. By applying the chain rule to the formula (11), we get

$$b'(x) - a''(\dot{x}) \cdot \ddot{x} = 0. \quad (12)$$

We want to find a Lagrangian that leads to differential equation (9). For this Lagrangian, the formula (12) will be true when we substitute the expression (9) for the acceleration \ddot{x} . As a result, we get the following formula

$$b'(x) + k_0 \cdot a''(\dot{x}) \cdot \dot{x} = 0, \quad (13)$$

i.e., equivalently,

$$k_0 \cdot a''(\dot{x}) \cdot \dot{x} = -b'(x) \quad (14)$$

for all possible values x and \dot{x} .

The left-hand side of the formula (14) does not depend on \dot{x} , and its right-hand side does not depend on x . Since these two sides are equal, this means that this expression cannot depend neither on x nor on \dot{x} and is, therefore, a constant. Let us denote this constant by C . Then, from the condition that the right-hand side is equal to this constant, we conclude that $b'(x) = -C$, hence $b(x) = -C \cdot x + C_0$. The constant term C_0 in the Lagrangian does not affect the corresponding equations (5) and can thus be safely ignored. So, we have $b(x) = -C \cdot x$.

Similarly, from the condition that the left-hand side of the formula (14) is equal to the constant C , we conclude that

$$k_0 \cdot a''(y) \cdot y = C, \tag{15}$$

where, for simplicity, we denoted $y \stackrel{\text{def}}{=} \dot{x}$. From (15), we conclude that

$$a''(y) = \frac{C}{k_0 \cdot y}. \tag{16}$$

Integrating over y , we get

$$a'(y) = \frac{C}{k_0} \cdot \ln(y) + C_0, \tag{17}$$

and, integrating once again, that

$$a(y) = \frac{C}{k_0} \cdot y \cdot \ln(y) + C_0 \cdot y + C_1. \tag{18}$$

Ignoring the constant C_1 and taking into account that $L(x, \dot{x}) = a(\dot{x}) + b(x)$ and that $b(x) = -C \cdot x$, we get the following expression for the desired Lagrangian:

Resulting Lagrangian. The system (9) can be described by the Lagrangian

$$L(x, \dot{x}) = \frac{C}{k_0} \cdot \dot{x} \cdot \ln(\dot{x}) + C_0 \cdot \dot{x} - C \cdot x. \tag{19}$$

Comment. One can easily check that for this Lagrangian, Euler-Lagrange equations (5) indeed lead to the equations (9).

Resulting expression for conserved “energy”. Here,

$$\frac{\partial L}{\partial \dot{x}} = \frac{C}{k_0} \cdot (\ln(\dot{x}) + 1) + C_0.$$

Thus, applying the usual formula (7) to the Lagrangian (19), we get the expression

$$E_L = \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} - L = \frac{C}{k_0} \cdot \dot{x} + C \cdot x. \tag{20}$$

One can easily check that this “energy” is indeed conserved. Indeed, here

$$\frac{dE_L}{dt} = \frac{d}{dt} \left(\frac{C}{k_0} \cdot \dot{x} + C \cdot x \right) = \frac{C}{k_0} \cdot \ddot{x} + C \cdot \dot{x}. \tag{21}$$

Substituting the expression $\ddot{x} = -k_0 \cdot \dot{x}$ into this formula, we indeed get $\frac{dE_L}{dt} = 0$.

3. From the Simplest Example to a General Dynamical System

What we do in this section. One may think that the weird conclusion — that for a friction particle, energy is well-defined and conserved — is caused by the fact that we have selected a very simple dynamical system (1). Alas, this is not the case. Let us show that a similar Lagrangian reformulation is possible for a generic dynamical system

$$\ddot{x}_i = f_i(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n), \quad i = 1, \dots, n. \quad (22)$$

A simple multi-D case. Let us start with a multi-D analog of a system with friction, in which the differential equations have the form

$$\ddot{x}_i(t) = -k_0 \cdot \dot{x}_i(t). \quad (23)$$

This system can be described, e.g., by a Lagrangian

$$L = \sum_i \frac{1}{k_0} \cdot \dot{x}_i \cdot \ln(\dot{x}_i) - \sum_i x_i. \quad (24)$$

General case. In the general case, differential equations (5) take the form

$$L_{,x_i} - \frac{d}{dt} L_{,\dot{x}_i} = 0, \quad (25)$$

where $L_{,z}$ denotes partial derivative. By using the chain rule to differentiate the expression $L_{,\dot{x}_i}(x_j, \dot{x}_j)$, we get

$$L_{,x_i} - \sum_j L_{,\dot{x}_i x_j} \cdot \dot{x}_j - \sum_j L_{,\dot{x}_i \dot{x}_j} \cdot \ddot{x}_j. \quad (26)$$

Substituting $\ddot{x}_i = f_i$ into this formula and using notations $y_i = \dot{x}_i$, we get

$$L_{,x_i} - \sum_j L_{,y_i x_j} \cdot y_j - \sum_j L_{,y_i y_j} \cdot f_j(x_1, \dots, x_n, y_1, \dots, y_n). \quad (27)$$

Our objective is to define a function $L(x_1, \dots, x_n, y_1, \dots, y_n)$ of $2n$ variables for which the second-order partial differential equation (27) holds.

Let us show how we can construct such a function. Let us take, e.g., $L(x_1, \dots, x_n, 0, \dots, 0) = 0$ when all the derivatives y_i are equal to 0. Then, we extend it to the case when $y_1 \neq 0$ and $y_2 = \dots = y_n = 0$. With respect to y_1 , (27) becomes a simple second order equation

$$\frac{\partial^2 L}{\partial y_1^2} \cdot f_1 + \frac{\partial^2 L}{\partial y_1 \partial \dots} \cdot \dots + \dots,$$

from which one can explicitly obtain such an extension — e.g., by Euler-style step-by-step integration. Then, we can extend this function along y_2 , etc. At the end, we get a function defined for all possible values of x_i and y_j .

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REFERENCES

1. Feynman R., Leighton R., and Sands M. The Feynman lectures on physics, Boston, Massachusetts : Addison Wesley, 2005.
2. Landau L.D. and Lifschitz, E.M. The Classical Theory of Fields, The Classical Theory of Fields, Oxford, UK : Butterworth-Heinemann, 1980.