WIENER-PROCESS-TYPE EVASIVE AIRCRAFT ACTIONS ARE INDEED OPTIMAL AGAINST ANTI-AIRCRAFT GUNS: WIENER'S DATA REVISITED

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Abstract. In his 1940s empirical study of evasive aircraft actions, N. Wiener, the father of cybernetics, found out that the pilot's actions follow a Wiener-type-process. In this paper, we explain this empirical result by showing that such evasive actions are indeed optimal against the 1940s anti-aircraft guns.

Keywords: Wiener process, N. Wiener's experiments, evasive aircraft maneuvers, optimization.

1. Introduction

Wiener's empirical data. Many techniques that form the basis of modern communication, signal processing, and control were developed in the 1940s by MIT's Norbert Wiener, as part of his *cybernetics*. Cybernetics is where the now ubiquitous abbreviation "cyber" — ranging from cyberinfrastructure to cyberbullying — comes from. Wiener's work was boosted during the Second World War, when he worked on automatic control devices for anti-aircraft guns; see, e.g., [1, 4].

Wiener used statistical optimization techniques to develop a firing strategy that would, on average, be most efficient against the pilot's random evasive maneuvers. To formulate the corresponding optimization problem, it is necessary to know the probabilities of different evasive trajectories $\vec{x}(t)$. To find these probabilities, N. Wiener, with his collaborator Julian Bigelow, a pilot by training, set up a flight simulator and recorded the corresponding evasive trajectories.

As a result, for the simplified situation with no restriction of airplane maneuvering, the pilot's evasive trajectories followed the Browning motion (what is now known in Mathematics as a Wiener process) [3], when the change

$$\vec{x}(t + \Delta t) - \vec{x}(t)$$

can be in any spatial direction with equal probability, and the current change does not depend on the previous changes. In more realistic situations, when they took into account that the airplane' velocity $\vec{v}(t)$ cannot be changed abruptly, the change in velocities $\vec{v}(t + \Delta t) - \vec{v}(t)$ followed the Wiener process.

Comment.   Based on this information, N. Wiener and J. Bigelow developed an optimal controller. For this particular application, the resulting improvement in efficiency was very low, around a few percents, so this controller was not implemented. However, in many other applications, similar controllers were spectacularly successful.

Question. An interesting question is: why did the pilots use Wiener-process-type evasive actions? Are such evasive actions optimal — or are more efficient evasive maneuvers possible?

In this paper, we use simple game theory to show that the Wiener-process-type evasive actions are indeed optimal.

2.   Formulation of the Problem

Only 2-D coordinates are important. First, let us recall that from the viewpoint of the anti-aircraft gun, what is important is a 2-D location of an airplane: if the airplane travels in the 3-rd dimension, along the line of fire, it does not help it evade the shells. Because of this, we will only consider 2-D locations and velocities.

An idealized situation when an aircraft can arbitrarily change its velocity: towards the exact formulation of the problem. Let us first consider the simplified setting, when it is assumed that an aircraft can arbitrarily change its speed $\vec{v}(t)$, as long as this speed does not exceed the limit v_0 imposed by its engine. In this case, if at the moment t, the aircraft was at location $\vec{x}(t)$, by the next moment of time $t + \Delta t$ it can travel any distance not exceeding $v_0 \cdot \Delta t$. Thus, at the moment $t + \Delta t$, the aircraft can be anywhere in the disk D of radius $v_0 \cdot \Delta t$ centered at the point $\vec{x}(t)$. Selecting an evasive maneuver means selecting a probability distribution $\rho_p(\vec{x})$ on this disk, a distribution that determines with what probability the plane will be at a given location.

The adversary observes the position $\vec{x}(t)$ and the type of the plane, so the adversary knows the plane's maximum velocity v_0 and thus, knows the disk D of possible locations of the plane at the next moment of time. Once the disk is known, the adversary selects his own probability distribution $\rho_s(\vec{x})$, distribution that describes with what probability the shell is aimed towards a future location $\vec{x}$.

The goal of the pilot is to evade the shell, i.e., to minimize the probability of being hit, while the goal of the gunner is to hit the plane, i.e., to maximize this probability. A shell hits the plane if it is sufficiently close to the plane, i.e., if the position $\vec{x}_s$ of the shell is within a certain small distance $\varepsilon > 0$ from the position $\vec{x}_p$ of the plane. For each position $\vec{x}_s$ of the shell, the plane is hit if this plane is within a circle C of radius ε with a center in $\vec{x}_s$. The probability for a plane to be in this circle is equal to $\int_C \rho_p(\vec{y}) \, d\vec{y}$. Since the circle is small, the value $\rho_p(\vec{y})$ is practically constant within this circle, so this integral can be approximated as $A_\varepsilon \cdot \rho_p(\vec{x}_s)$, where $A_\varepsilon = \pi \cdot \varepsilon^2$ is the area of this circle.

For each location $\vec{x}_s$ of the shell, the probability of a plane being hit is thus equal to $A_\varepsilon \cdot \rho_p(\vec{x}_s)$. The probability of a shell being in this location is proportional

to $\rho_s(\vec{x}_s)$. Thus, by using the formula of complete probability, we can compute the probability of being hit as $A_\varepsilon \cdot \int_D \rho_s(\vec{x}) \cdot \rho_p(\vec{x}) \, d\vec{x}$.

This is a zero-sum game: a win for the plane — successful evasion of the shell — is a loss for the adversary. So, according to game theory (see, e.g., [2]), the optimal strategy for a pilot is a *minimax* strategy, i.e., a strategy that minimize the worst-case loss. For this strategy, the worst-case value

$$J(\rho_p) = A_\varepsilon \cdot \max_{\rho_s(\vec{x})} \int_D \rho_s(\vec{x}) \cdot \rho_p(\vec{x}) \, d\vec{x}$$

is the smallest possible. Let us show how to solve this optimization problem.

Solving the resulting problem. The above integral is the expected (mean) value of the probability density function $\rho_p(\vec{x})$ over the distribution $\rho_s(\vec{x})$. The expected value of any function is always smaller than or equal to the maximum of this function, so $\int_D \rho_s(\vec{x}) \cdot \rho_p(\vec{x}) \, d\vec{x} \le \max_{\vec{x} \in D} \rho_p(\vec{x})$. Thus,

$$\max_{\rho_s(\vec{x})} \int_D \rho_s(\vec{x}) \cdot \rho_p(\vec{x}) \, d\vec{x} \le \max_{\vec{x} \in D} \rho_p(\vec{x}).$$

On the other hand, if we take a distribution $\rho_s(\vec{x})$ which is located, with probability 1, at a point $\vec{x}$ where the function $\rho_p(\vec{x})$ attains its maximum, then we will get $\int_D \rho_s(\vec{x}) \cdot \rho_p(\vec{x}) \, d\vec{x} = \max_{\vec{x} \in D} \rho_p(\vec{x})$. Thus,

$$\max_{\rho_s(\vec{x})} \int_D \rho_s(\vec{x}) \cdot \rho_p(\vec{x}) \, d\vec{x} = \max_{\vec{x} \in D} \rho_p(\vec{x}),$$

and therefore the value $J(\rho_p)$ is equal to $A_\varepsilon \cdot \max_{\vec{x} \in D} \rho_p(\vec{x})$:

$$J(\rho_p) = A_\varepsilon \cdot \max_{\vec{x} \in D} \rho_p(\vec{x}).$$

Minimizing $J(\rho_p)$ is hence equivalent to minimizing the value $\max_{\vec{x} \in D} \rho_p(\vec{x})$. We know that $\int_D \rho_p(\vec{x}) \, d\vec{x} = 1$. Here, for every $\vec{x}$, we have $\rho_p(\vec{x}) \le m \stackrel{\text{def}}{=} \max_{\vec{x} \in D} \rho(\vec{x})$, thus, $1 = \int_D \rho(\vec{x}) \, d\vec{x} \le \int_D m \, d\vec{x} \le m \cdot A(D)$, where $A(D)$ is the area of the region D. From $1 \le m \cdot A(D)$, we conclude that $m \ge \dfrac{1}{A(D)}$. The equality is possible only when there is equality for all $\vec{x}$ in the inequality $\rho_p(\vec{x}) \le m$, i.e., when $\rho_p(\vec{x}) = m$ for all $\vec{x}$. This is exactly a uniform distribution on the set D — and of course, this distribution should be independent on what was done in the past.

Thus, in the simplified case, we indeed conclude that the Wiener process is an optimal way to perform evasive actions.

A more realistic formulation of the problem. A more realistic description of evasive actions must take into account that the velocity $\vec{v}(t + \Delta t)$ at the next moment of time $t + \Delta t$ cannot be too much different from the velocity $\vec{v}(t)$ at the previous moment of time, there is a limit on acceleration $|\vec{a}(t)| \le a_0$. In general,

once we know the initial location $\vec{x}(t)$, the initial velocity $\vec{x}(t)$, and the acceleration $\vec{a}(t)$, we can determine the position $\vec{x}(t + \Delta t)$ at the next moment of time as

$$\vec{x}(t + \Delta t) = \vec{x}(t) + \vec{v}(t) \cdot \Delta t + \frac{1}{2} \cdot \vec{a}(t) \cdot (\Delta t)^2.$$

Here, the initial location $\vec{x}(t)$ and the initial velocity $\vec{v}(t)$ are fixed, and the acceleration $\vec{a}(t)$ can take any value for which $|\vec{a}(t)| \leq a_0$.

Thus, the set of locations $\vec{x}(t+\Delta)$ is a disk D centered at the point $\vec{x}(t)+\vec{v}(t)\cdot\Delta t$ with radius $\frac{1}{2} \cdot a_0 \cdot (\Delta t)^2$. Similarly to the simplified case, we can describe possible evasive actions by a probability density $\rho_p(\vec{x})$ located on this disk, and, similarly to the simplified case, we can conclude that the optimal evasive action corresponds to the uniform distribution on this disk. In this optimal solution, the change in velocity $\vec{v}(t + \Delta t) - \vec{v}(t) = \vec{a}(t) \cdot \Delta t$ is uniformly distributed on the disk of radius $a_0 \cdot \Delta t$ – and is independent on the previous trajectory of the plane.

Thus, in this realistic case, we indeed conclude that the Wiener process for velocities is indeed an optimal way to perform evasive actions. So, Wiener's empirical data indeed corresponds to optimal evasive action.

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УКЛОНЕНИЕ САМОЛЁТА ОТ ЗЕНИТНЫХ ОРУДИЙ ПО ТИПУ ВИНЕРОВСКОГО ПРОЦЕССА ОПТИМАЛЬНО: НОВОЕ ОБЪЯСНЕНИЕ ДАННЫХ ВИНЕРА

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Аннотация. В 1940 году в эмпирическом исследовании уклонения авиации от зенитных орудий, Н. Винер, отец кибернетики, доказывает, что действия пилота должны выглядеть как винеровский случайный процесс. В этой статье мы объясним этот эмпирический результат, показав, что такие действия по уклонению действительно оптимальны против зенитных орудий 1940 года.

Ключевые слова: винеровский процесс, эксперимент Винера, манёвры уклонения, оптимизация.