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SOME MATH À LA ALEXANDROV

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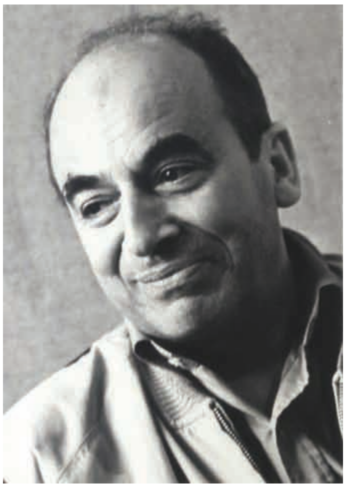
Abstract. This is a short overview of the impact and current state of Alexandrov's functional analytical approach to extremal problems in the space of convex syrfaces.

Keywords: A.D. Alexandrov, extreme problems, convex surfaces.

Alexandrov's Math Is Great and Alive



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The Mathematics Subject Classification, produced jointly by the editorial staffs of *Mathematical Reviews* and *Zentralblatt für Mathematik* in 2020, has Section 53C45 "Global surface theory (convex surfaces à la A. D. Aleksandrov)".

Good mathematics starts as a first love. If great, it turns into adult sex and happy marriage. If ordinary, it ends in dumping, cheating, or divorce. If awesome, it becomes eternal. Alexandrov's mathematics is great. To demonstrate, inspect his solution of the Minkowski problem.

Alexandrov's mathematics is alive, expanding and flourishing for decades. Dido's problem and its present-day next of kin is one of the examples we will elaborate below.

The Space of Convex Bodies

A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in R^N and its *support function* $S(z) := \sup\{(x,z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set $\mathscr{V}_N$ of all compact convex subsets of $\mathbb{R}^N$.

The Minkowski duality makes $\mathscr{V}_N$ into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. The *linear span* $[\mathscr{V}_N]$ of $\mathscr{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*.

The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 [1] and the further insights of Radström, Hörmander, and Pinsker [2].

Alexandrov Measures

Alexandrov proved the unique existence of a translate of a convex body given its surface area function, thus completing the solution of the Minkowski problem. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons.

Each Alexandrov measure is a translation-invariant additive functional over the cone $\mathscr{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathscr{A}_N$.

Blaschke's Sum

Given $\mathfrak{x}, \mathfrak{y} \in \mathscr{V}_N$, the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ means that $\mathfrak{x}$ and y are equal up to translation or, in other words, are translates of one another. So, $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geqslant_{\mathbb{R}^N}$ on $\mathscr{V}_N$ of the possibility of inserting one figure into the other by translation.

The sum of the surface area measures of $\mathfrak{x}$ and $\mathfrak{y}$ generates the unique class $\mathfrak{x}\#\mathfrak{y}$ of translates which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

There is no need in discriminating between a convex figure, the coset of its translates in $\mathscr{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathscr{A}_N$.

Comparison Between the Structures

Objects	Minkowski	Blaschke
cone of sets	$\mathscr{V}_N/\mathbb{R}^N$	$\mathscr{A}_N$
dual cone	$\mathscr{V}_N^*$	$\mathscr{A}_N^*$
positive cone	$\mathscr{A}_N^*$	$\mathscr{A}_N$
linear functional	$V_1(\mathfrak{z}_N, \cdot)$, breadth	$V_1(\cdot, \mathfrak{z}_N)$, area
concave functional	$V^{1/N}(\cdot)$	$V^{(N-1)/N}(\cdot)$
convex program	isoperimetric problem	Urysohn's problem
operator constraint	inclusion-like	curvature-like
Lagrange's multiplier	surface	function
gradient	$V_1(\bar{\mathfrak{x}}, \cdot)$	$V_1(\cdot, \bar{\mathfrak{x}})$

The Natural Duality

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Let $[\mathscr{A}_N]$ be the space $\mathscr{A}_N - \mathscr{A}_N$ of translation-invariant measures, in fact, the linear span of the set of Alexandrov measures.

$C(S_{N-1})/\mathbb{R}^N$ and $[\mathscr{A}_N]$ are made dual by the canonical bilinear form

$$\langle f, \mu\rangle = \frac{1}{N}\int_{S_{N-1}} f d\mu$$
$$(f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathscr{A}_N]).$$

For $\mathfrak{x} \in \mathscr{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathscr{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y}\rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$.

Solution of Minkowski's Problem

Alexandrov observed that the gradient of $V(\cdot)$ at $\mathfrak{x}$ is proportional to $\mu(\mathfrak{x})$ and so minimizing $\langle \cdot, \mu\rangle$ over $\{V = 1\}$ will yield the equality $\mu = \mu(\mathfrak{x})$ by the Lagrange multiplier rule. But this idea fails since the interior of $\mathscr{V}_N$ is empty. The fact that DC-functions are dense in $C(S_{N-1})$ is not helpful at all.

Alexandrov extended the volume to the positive cone of $C(S_{N-1})$ by the formula $V(f) := \langle f, \mu(\mathrm{co}(f))\rangle$ with $\mathrm{co}(f)$ the envelope of support functions below f. The ingenious trick settled all!

This was done by Alexandrov in 1938 but still is one of the summits of convexity.

Is Dido's Problem Solved?

From a utilitarian standpoint, the answer is definitely in the affirmative. There is no evidence that Dido experienced any difficulties, showed indecisiveness, and procrastinated the choice of the tract of land. Practically speaking, the situation in which Dido made her decision was not as primitive as it seems at the first glance. The appropriate generality is unavailable in the mathematical model known as the classical isoperimetric problem.

Dido's problem inspiring our ancestors remains the same intellectual challenge as Kant's starry heavens above and moral law within.

Pareto Optimality

Consider a bunch of economic agents each of which intends to maximize his own income. The *Pareto efficiency principle* asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

Vector Isoperimetric Problem

Given are some convex bodies $\mathfrak{y}_1, \dots, \mathfrak{y}_M$. Find a convex body $\mathfrak{x}$ encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \mathfrak{y}_1), \dots, V_1(\mathfrak{x}, \mathfrak{y}_M)$. In symbols,

$$\mathfrak{x} \in \mathscr{A}_N;\ \widehat{p}(\mathfrak{x}) \geqslant \widehat{p}(\bar{\mathfrak{x}});\ (\langle \mathfrak{y}_1, \mathfrak{x}\rangle, \dots, \langle \mathfrak{y}_M, \mathfrak{x}\rangle) \to \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure.

Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form

$$\bar{\mathfrak{x}} = \alpha_1 \mathfrak{y}_1 + \dots + \alpha_m \mathfrak{y}_m,$$

where $\alpha_1, \dots, \alpha_m$ are positive reals.

The Leidenfrost Problem

Given the volume of a three-dimensional convex figure, minimize its surface area and vertical breadth.

By symmetry everything reduces to an analogous plane two-objective problem, whose every Pareto-optimal solution is by 2 a *stadium*, a weighted Minkowski sum of a disk and a horizontal straight line segment.

A plane spheroid, a Pareto-optimal solution of the Leidenfrost problem, is the result of rotation of a stadium around the vertical axis through the center of the stadium.

Internal Urysohn Problem with Flattening

Given are some convex body $\mathfrak{x}_0 \in \mathscr{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \subset \mathfrak{x}_0$ of fixed integral breadth, maximize the volume of $\mathfrak{x}$ and minimize the breadth of $\mathfrak{x}$ in the flattening direction: $\mathfrak{x} \in \mathscr{V}_N$; $\mathfrak{x} \subset \mathfrak{x}_0$; $\langle \mathfrak{x}, \mathfrak{z}_N\rangle \geqslant \langle \bar{\mathfrak{x}}, \mathfrak{z}_N\rangle$; $(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf$.

For a feasible convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction $\bar{z}$ it is necessary and sufficient that there be positive reals α, β and a convex figure $\mathfrak{x}$ satisfying

$$\mu(\bar{\mathfrak{x}}) = \mu(\mathfrak{x}) + \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});$$
$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{spt}(\mu(\mathfrak{x})).$$

Here $\operatorname{spt}(\mu)$ is the support of a measure μ.

Rotational Symmetry

Assume that a plane convex figure $\mathfrak{x}_0 \in \mathscr{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator $\bar{z}$. Assume further that $\mathfrak{x}_{00}$ is the result of rotating $\mathfrak{x}_0$ around the symmetry axis $A_{\bar z}$ in $\mathbb{R}^3$.

$$\begin{gathered}\mathfrak{x} \in \mathscr{V}_3;\\ \mathfrak{x} \text{ is a convex body of rotation around } A_{\bar z};\\ \mathfrak{x} \supset \mathfrak{x}_{00};\ \langle \mathfrak{z}_N, \mathfrak{x}\rangle \geqslant \langle \mathfrak{z}_N, \bar{\mathfrak{x}}\rangle;\\ (-p(\mathfrak{x}), b_{\bar z}(\mathfrak{x})) \to \inf.\end{gathered}$$

Each Pareto-optimal solution is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.

Soap Bubbles

Little is known about the analogous problems in arbitrary dimensions. An especial place is occupied by the result of Porogelov who demonstrated that the "soap bubble" in a tetrahedron has the form of the result of the rolling of a ball over a solution of the internal Urysohn problem, i.e. the weighted Blaschke sum of a tetrahedron and a ball.

The External Urysohn Problem with Flattening

Given are some convex body $\mathfrak{x}_0 \in \mathscr{V}_N$ and flattening direction $\bar z \in S_{N-1}$. Considering $\mathfrak{x} \supset \mathfrak{x}_0$ of fixed integral breadth, maximize volume and minimizing breadth in the flattening direction: $\mathfrak{x} \in \mathscr{V}_N;\ \mathfrak{x} \supset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N\rangle \geqslant \langle \bar{\mathfrak{x}}, \mathfrak{z}_N\rangle;\ (-p(\mathfrak{x}), b_{\bar z}(\mathfrak{x})) \to \inf.$

For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening it is necessary and sufficient that there be positive reals α, β, and a convex figure $\mathfrak{x}$ satisfying

$$\begin{gathered}\mu(\bar{\mathfrak{x}}) + \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar z} + \varepsilon_{-\bar z});\\ V(\bar{\mathfrak{x}}) + V_1(\mathfrak{x}, \bar{\mathfrak{x}}) = \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar z}(\bar{\mathfrak{x}});\\ \bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{spt}(\mu(\mathfrak{x})).\end{gathered}$$

Optimal Convex Hulls

Given $\mathfrak{y}_1, \dots, \mathfrak{y}_m$ in $\mathbb{R}^N$, place $\mathfrak{x}_k$ within $\mathfrak{y}_k$, for $k := 1, \dots, m$, maximizing the volume of each of the $\mathfrak{x}_1, \dots, \mathfrak{x}_m$ and minimize the integral breadth of their convex hull:

$$\mathfrak{x}_k \subset \mathfrak{y}_k;\ (-p(\mathfrak{x}_1), \dots, -p(\mathfrak{x}_m), \langle \operatorname{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\}, \mathfrak{z}_N\rangle) \to \inf.$$

For some feasible $\bar{\mathfrak{x}}_1, \dots, \bar{\mathfrak{x}}_m$ to have a Pareto-optimal convex hull it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_m \in \mathbb{R}_+$ not vanishing simultaneously and positive Borel measures $\mu_1, \dots, \mu_m$ and $\nu_1, \dots, \nu_m$ on S_{N-1} such that

$$\begin{gathered}\nu_1 + \dots + \nu_m = \mu(\mathfrak{z}_N);\\ \bar{\mathfrak{x}}_k(z) = \mathfrak{y}_k(z) \quad (z \in \operatorname{spt}(\mu_k));\\ \alpha_k\mu(\bar{\mathfrak{x}}_k) = \mu_k + \nu_k\ (k := 1, \dots, m).\end{gathered}$$

See [3] and [4] for more detail.

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НЕМНОГО МАТЕМАТИКИ А-ЛЯ АЛЕКСАНДРОВ

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Аннотация. Это краткий обзор влияния и текущего состояния функционально-аналитического подхода Александрова к экстремальным задачам в пространстве выпуклых поверхностей.

Ключевые слова: А.Д. Александров, экстремальные задачи, выпуклые поверхности.

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