

ORDER RELATIONS ARE UBIQUITOUSLY FUNDAMENTAL: ALEXANDROV(–ZEEMAN) THEOREM EXTENDED FROM SPACE-TIME PHYSICS TO LOGICAL REASONING AND DECISION MAKING

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Abstract. In all areas of human activity, there are natural ordering relations: causality in space-time physics, preference in decision making, and logical inference in reasoning. In space-time physics, a 1950 theorem by A.D. Alexandrov proved that causality relation is fundamental: many other features, including numerical characteristics of time and space, can be reconstructed from this relation. In this paper, we provide simple proofs that, similarly, the corresponding ordering relations are fundamental in decision making and in logical reasoning.

Keywords: order relations, Alexandrov-Zeeman Theorem, space-time physics, decision making, preference relation, logical reasoning, fuzzy logic, implication operation.

1. Order Relations Are Important



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Main objectives of science and engineering. One of the main objectives of science and engineering is to help people *select the most beneficial decisions*. To make these decisions,

- we must know people's *preferences*,
- we must have the information about different events
 - *possible consequences of different decisions*,
 - and
- we must be able to use this information to come up with decisions.

Enter order relations. In each of these three categories, we have natural order relations:

- for preferences, $a \leq b$ means that b is preferable to a ;
- for events, $a \leq b$ means that a can influence b ; this relation is known as causality; and
- in reasoning, $a \leq b$ means that we can infer b from a ; this relation is known as *implication* and is usually denoted by $a \rightarrow b$.

Important comment: these relations are often not binary. Sometimes, we are absolutely sure that an alternative b is better than an alternative a , that an event a can influence an event b , that a statement a definitely implies the statement b . In such cases, the corresponding relation is “binary” in the sense that:

- for some pairs (a, b) , the relation $a \leq b$ is absolutely true, while
- for some other pairs (a, b) , the relation $a \leq b$ is absolutely *not* true.

However, in many cases, we are not 100% certain:

- we believe that the alternative b is most probably better, but we have some doubts; as a result, people sometimes select a ;
- we believe that b can probably be inferred from a , but we are not absolutely sure; for example, in a trial by jury, there is often a reasonable doubt, and, because of this, a suspect goes free in spite of some strong indirect evidence against him.

In the following text, we will take this uncertainty into account.

What we do in this paper. It is known that the causality relation is fundamental in space-time physics, in the sense that many other properties are uniquely determined by this relation. This fact was first proved by A. D. Alexandrov in 1950.

In this paper, we provide simple proofs that ordering relations are – in this sense – fundamental in decision making and logical reasoning as well.



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2. Alexandrov-Zeeman Theorem: Reminder

Causality in physics: a brief reminder. To describe the Alexandrov-Zeeman theorem, let us first briefly recall how causality is defined in physics; see, e.g., [10, 43].

In Newton's physics, signals can potentially travel with an arbitrarily large speed. To describe the corresponding causality relation between events, let us denote an event occurring at the spatial location x at time t by $a = (t, x)$. In these notations, Newton's causality relation is as follows: an event $a = (t, x)$ can causally (physically) influence an event $a' = (t', x')$ if and only if $t \leq t'$:

$$(t, x) \leq (t', x') \Leftrightarrow t \leq t'. \quad (1)$$



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In Special Relativity, the speed of all the signals is limited by the speed of light c . As a result, $a = (t, x) \leq a' = (t', x')$ if and only if $t' \geq t$ and in time $t' - t$, the speed needed to traverse the distance $d(x, x')$ does not exceed c , i.e., $\frac{d(x, x')}{t' - t} \leq c$. The resulting causality relation has the form

$$(t, x) \leq (t', x') \Leftrightarrow c \cdot (t' - t) \geq d(x, x'). \quad (2)$$

Alexandrov-Zeeman theorem. In 1950, A. D. Alexandrov showed that in Special Relativity, causality implied Lorenz group [1,2]. To be more precise, he proved that every transformation of the 4-dimensional space-time that preserves the causality relation (2) is linear, and is a composition of:

- shifts in space and time,
- spatial rotations,
- Lorentz transformations (describing a transition to a moving reference frame), and
- re-scalings $x \rightarrow \lambda \cdot x$ (corresponding to a change of unit for measuring space and time).

This theorem was later generalized by E. Zeeman [3] and is therefore known as the *Alexandrov-Zeeman theorem*; see, e.g., [3,3–5,7,8,13–18,20,21,23–29,36,41].

This theorem showed that causality indeed plays a fundamental role in space-time physics: once we know this relation, we can reconstruct the linear structure of space-time, we can reconstruct (modulo a possible multiplicative constant) the values of proper time and proper space, etc.

3. Order Relation in Decision Making

How preferences are described in decision theory: a brief reminder. We want to prove that the notion of preference is as fundamental in decision making as

causality is in space-time physics. For this purpose, let us recall how preferences are described in decision theory; see, e.g., [11, 12, 22, 31, 37, 38, 42].

The usual way to provide a numerical description of preferences is to select two alternatives:

- a very bad alternative A_- which is worse than anything we will actually encounter, and
- a very good alternative A_+ which is better than anything we will actually encounter.

Then, to provide a numerical value to any actual alternative A , we ask the person to compare this alternative with lotteries in which this person:

- gets A_+ with some probability p and
- gets A_- with the remaining probability $1 - p$,

for different values p . We will denote such a lottery by $L(p)$.

- When the probability p is small, the lottery is almost the same as the very bad alternative A_- . So, due to our choice of A_- , the alternative A is better than $L(p)$: $L(p) \leq A$.
- When the probability p is close to 1, the lottery is almost the same as the very good alternative A_+ . So, due to our choice of A_+ , the alternative A is worse than $L(p)$: $A \leq L(p)$.

As we increase the probability p of the very good outcome A_+ , the lottery becomes more and more preferable. So, at some probability p_0 , the decision maker switches from $L(p) < A$ to $A < L(p)$. This threshold value p_0 is known as the *utility* of the alternative A . The utility is usually denoted by $u(A)$.

Utility is not uniquely determined. The numerical value of the utility depends on our choice of the alternatives A_- and A_+ . One can show that if we select a different pair A'_-, A'_+ , then the corresponding numerical value $u'(A)$ is related to the original value $u(A)$ by a linear dependence

$$u'(A) = a \cdot u(A) + b, \tag{3}$$

for some values $a > 0$ and b (which do not depend on the alternative A).

How people actually make decisions. In the ideal world, if a person encounters n alternatives with utilities u_1, \dots, u_n , this person should select the alternative with the largest utility – because, by definition of utility, this alternative is equivalent to the lottery in which the probability of winning the big prize A_+ is the highest. In reality, the person may select other alternatives as well. The probability of selecting the i -th is equal to:

$$p_i = \frac{\exp(\alpha \cdot u_i)}{\sum_{j=1}^n \exp(\alpha \cdot u_j)}, \tag{4}$$

for some value $\alpha > 0$ depending on the person. This formula is known as the *discrete choice model*; see, e.g., [30, 32–34, 44]. For this formula, D. McFadden received a Nobel Prize.

The formula (4) confirms what we have mentioned earlier: that human preference is not a binary relation, it is characterized by the probabilities p_i of selecting different alternatives.

Comment. As shown in [9], the formula (4) is not just empirically valid: it can be explained by natural symmetry ideas.

Main result of this section. The following result shows that the preference relation – as described by the probabilities p_i – determines utilities uniquely – modulo the linear transformation (3). Thus, for decision making, the corresponding ordering relation is indeed also fundamental.

Proposition 1. *If for two sequences u_1, \dots, u_n and u'_1, \dots, u'_n and for some values $\alpha > 0$ and $\alpha' > 0$, we have $p_i = p'_i$ for all i , where*

$$p_i \stackrel{\text{def}}{=} \frac{\exp(\alpha \cdot u_i)}{\sum_{j=1}^n \exp(\alpha \cdot u_j)} \quad \text{and} \quad p'_i = \frac{\exp(\alpha' \cdot u'_i)}{\sum_{j=1}^n \exp(\alpha' \cdot u'_j)}, \quad (5)$$

then there exists values $a > 0$ and b for which

$$u'_i = a \cdot u_i + b \quad (6)$$

for all i .

Proof. For each $i \neq 1$, if we divide the equality $p_i = p'_i$ by the equality $p_1 = p'_1$, we get

$$\frac{p_i}{p_1} = \frac{p'_i}{p'_1}. \quad (8)$$

Substituting the expressions (6) instead of p_i , p_1 , p'_i , and p'_1 , we conclude that

$$\frac{\exp(\alpha \cdot u_i)}{\exp(\alpha \cdot u_1)} = \frac{\exp(\alpha' \cdot u'_i)}{\exp(\alpha' \cdot u'_1)}, \quad (9)$$

i.e., equivalently,

$$\exp(\alpha \cdot (u_i - u_1)) = \exp(\alpha' \cdot (u'_i - u'_1)). \quad (10)$$

By taking logarithm of both sides and dividing both sides by α' , we conclude that

$$u'_i - u'_1 = a \cdot (u_i - u_1), \quad (11)$$

where we denoted $a \stackrel{\text{def}}{=} \alpha/\alpha'$. Hence,

$$u'_i = a \cdot u_i + (u'_1 - a \cdot u_1), \quad (12)$$

i.e., the desired formula (6) for $b \stackrel{\text{def}}{=} u'_1 - a \cdot u_1$.

We are almost done: we have proved the formula (6) for all $i \neq 1$. However, one can easily show that for $i = 1$, the right-hand side of the formula (12) is also equal to its left-hand side u'_1 . So, the proposition is proven.

4. Order Relation in Logical Reasoning

How can we describe degree of inference in logical reasoning: a brief reminder. A direction of logic that describes degrees of certainty – i.e., that analyzes statements which are imprecise (“fuzzy”) is known as *fuzzy logic*; see, e.g., [6, 19, 35, 39, 40, 45].

The original – and simplest – idea is to take into account that in a computer:

- “false” is represented as 0, and
- “true” is represented as 1.

Thus, it is reasonable to describe intermediate degrees by numbers from the interval $(0, 1)$.

There are many different *implication operations*, i.e., functions $f_{\rightarrow}(a, b)$ that transform the expert’s degree of confidence a and b in some statements A and B into the estimated degree of confidence in the implication $A \rightarrow B$. Such operation should satisfy several reasonable properties. For example:

- If we accept a statement A in which we are not 100% sure, then we can get conclusions that we could not get before. In this case, the statement $A \rightarrow B$ can have larger degree of confidence than the statement B itself.
- However, if, as A , we take an absolutely true statement T , with degree $a = 1$, then adding this statement will not change what we can conclude. Thus, our degree of confidence $f_{\rightarrow}(1, b)$ in the implication $T \rightarrow B$ should be exactly the same as our degree of confidence b in the original statement B : $f_{\rightarrow}(1, b) = b$.

The values $f_{\rightarrow}(a, b)$ corresponding to different a and b represent the inference ordering.

Main result of this section. The following result shows that the inference relation – as described by the degrees $f_{\&}(a, b)$ – uniquely determines the original degrees a and b .

Thus, for logical reasoning, the corresponding ordering relation is indeed also fundamental.

Definition 1. *By an implication operation, we mean a function*

$$f_{\rightarrow} : [0, 1] \times [0, 1] \rightarrow [0, 1] \tag{13}$$

for which $f_{\rightarrow}(1, b) = b$ for all b .

Proposition 2. *Suppose that we have two sequences a_1, \dots, a_n and a'_1, \dots, a'_n such that for some i_0 , we have $a_{i_0} = a'_{i_0} = 1$. Suppose also that we have two implication operations f_{\rightarrow} and f'_{\rightarrow} for which, for all i and j , we have*

$$f_{\rightarrow}(a_i, a_j) = f'_{\rightarrow}(a'_i, a'_j). \tag{14}$$

Then, for all j , we have $a_j = a'_j$.

Proof. For each j from 1 to n , substituting $i = i_0$ into the formula (14) and taking into account that $a_{i_0} = a'_{i_0} = 1$, we conclude that $f_{\rightarrow}(1, a_j) = f'_{\rightarrow}(1, a'_j)$. Now, by definition of an implication operation, we conclude that indeed $a_j = a'_j$.

The proposition is proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

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**ОТНОШЕНИЯ ПОРЯДКА ВСЕГДА ФУНДАМЕНТАЛЬНЫ:
ТЕОРЕМА АЛЕКСАНДРОВА(–ЗИМАНА), ПРОДОЛЖЕННАЯ С ФИЗИКИ
ПРОСТРАНСТВА-ВРЕМЕНИ НА ЛОГИЧЕСКИЕ РАССУЖДЕНИЯ
И ПРИНЯТИЕ РЕШЕНИЙ**

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Аннотация. Во всех сферах человеческой деятельности существуют естественные упорядочивающие отношения: причинность в физике пространства-времени, предпочтение в принятии решений и логический вывод в рассуждениях. В физике пространства-времени в теореме А.Д. Александрова 1950 года доказано, что отношение причинности является фундаментальным: многие другие характеристики, включая числовые характеристики времени и пространства, могут быть восстановлены из этого отношения. В этой статье мы приводим простые доказательства того, что и соответствующие упорядочивающие отношения являются основополагающими при принятии решений и логических рассуждений

Ключевые слова: отношения порядка, теорема Александрова–Зимана, физика пространства-времени, принятие решений, отношение предпочтений, логические рассуждения, нечёткая логика, импликационная операция.

Дата поступления в редакцию: 22.05.2021