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DISCRETE CAUSALITY IMPLIES LORENZ GROUP: CASE OF 2-D SPACE-TIMES

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Abstract. It is known that for Minkowski space-times of dimension larger than 2, any causality-preserving transformation is linear. It is also known that in a 2-D space-time, there are many nonlinear causality-preserving transformations. In this paper, we show that for 2-D space-times, if we restrict ourselves to discrete space-times, then linearity is retained: only linear transformation preserve causality.

Keywords: causality, special relativity, Alexandrov-Zeeman theorem, discrete space-time.

1. Formulation of the Problem

Causality in Special Relativity: a brief reminder. According to Special Relativity Theory, nothing can travel faster than the speed of light c. Thus, an event $a = (a_0, a_1, \ldots, a_n)$ that happens at moment a_0 at a spatial location $(a_1, \ldots, a_n)$ can affect the event $b = (b_0, b_1, \ldots, b_n)$ (we will denote it by $a \leqslant b$) if and only if a signal emitted by the first event can reach the second event by traveling at a speed not exceeding the speed of light, i.e., if and only if

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$$a \leqslant b \Leftrightarrow c \cdot (b_0 - a_0) \geqslant \sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2}.$$

This formula becomes simpler if we use the same units for time and distance, e.g., if we measure distance in light seconds or if we measure time in meters divided by c. In these units, the speed of light becomes 1, and the formula for the causal relation $a \leqslant b$ takes the following simplified form:

$$a \leqslant b \Leftrightarrow b_0 - a_0 \geqslant \sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2}. \quad (1)$$

Causality implies Lorentz group: general result. It is known that when the dimension n of proper space is at least 2, then any bijection of the $(n+1)$-dimensional space-time that preserves causality is linear, and it is a composition of rotations, shifts, Lorentz transformations, scalings $a \to \lambda \cdot a$, and discrete reflections; see, e.g., [1–3].

2-D space-time is an exception. The above result is valid for space-times of dimensions larger than 2. In the 2-D space-time, when $n = 1$, there are many non-linear transformations that preserve causality. They can be easily described if we take into account that for $n = 1$, the relation (1) – which, in this case, takes the form

$$a \leqslant b \Leftrightarrow b_0 - a_0 \geqslant |b_1 - a_1|, \tag{2}$$

can be describe in an even simpler form

$$a \leqslant b \Leftrightarrow (a_- \leqslant b_- \,\&\, a_+ \leqslant b_+),$$

where $a_- \stackrel{\text{def}}{=} a_0 - a_1$, $a_+ \stackrel{\text{def}}{=} a_0 + a_1$, and b_- and b_+ are defined similarly. From this description, it is clear that for any two strictly increasing bijections of real line f_- and f_+ (not necessarily linear ones) the transformation $(a_-, a_+) \mapsto (f_-(a_-), f_+(a_+))$ preserves causality.

What if space-time is discrete? A natural question is: what if, in the 2-D case, both space and time are discrete, i.e., what if there exists a “quantum” of space-time, and we can only have the values temporal and spatial coordinates a_0 and a_1 proportional to this quantum? If we select this quantum as a measuring unit, this means that both values a_0 and a_1 can only take integer values. In this case, what are transformations preserving the causal relation (2)?

In this paper, we prove that in this case, any bijection preserving causality is linear.

2. Definitions and the Main Result

Definition.

- *By a* causal relation *on the set M of all pairs of integers (a_0, a_1), we mean the relation* (2).
- *We say that a bijection $f : M \mapsto M$* preserves causality *if for all $a, b \in M$, we have $a \leqslant b$ if and only if $f(a) \leqslant f(b)$.*

Proposition. *A bijection preserves causality if and only if it is either a shift or a composition of a shift and spatial reflection $(a_0, a_1) \mapsto (a_0, -a_1)$.*

Proof.

1°. It is easy to see that a shift and spatial reflection both preserve causality. So, to complete our proof, it is sufficient to prove that every bijection that preserves causality has the desired form. So, let F be such a bijection.

2°. Let us denote $F((0,0))$ by (b_0, b_1). Then, the composition f of the original bijection F and a shift $(a_0, a_1) \mapsto (a_0 - b_0, a_1 - b_1)$ also preserves causality, and it transforms the point $(0,0)$ into itself. A composition of two causality-preserving transformations is also causality-preserving.

Once we know the transformation f, we can reconstruct the original transformation F as a composition of F and the opposite shift

$$(a_0, a_1) \mapsto (a_0 + b_0, a_1 + b_1).$$

Thus, if we prove that the composition f is either a shift or a composition of a shift and a reflection, then the same is true for the original transformation F.

So, it is sufficient to prove the proposition for transformations that transform $(0,0)$ into $(0,0)$. Hence, without losing generality, we will assume that

$$f((0,0)) = (0,0).$$

3°. Let us, as usual, define the strict order relation $a < b$ as $(a \leqslant b) \,\&\, (a \neq b)$. Let us prove that if $a < b$, then $a_0 < b_0$.

Indeed, by (2), if $a \leqslant b$, then $a_0 \leqslant b_0$. If $a_0 = b_0$, then the formula (2) implies that $a_1 = b_1$, thus $a = b$. So, if $a < b$, then we indeed have $a_0 < b_0$, and thus, that

$$b_0 \geqslant a_0 + 1.$$

4°. Let us define the "immediately precedes" relation as follows:

$$a \prec b \Leftrightarrow (a < b \,\&\, \neg \exists c\, (a < c < b)).$$

Let us prove that

$$a \prec b \Leftrightarrow (b_0 - a_0 = 1 \,\&\, |a_1 - b_1| \leqslant 1).$$

4.1°. Let us first prove that if $b_0 - a_0 = 1$ and $|a_1 - b_1| \leqslant 1$, then $a \prec b$.

In this case, the fact that $a \leqslant b$ follows directly from the formula (2), so all we need to prove is that there is no event c for which $a < c < b$. Indeed, if such event c existed, then, due to Part 3 of this proof, we would have $b_0 \geqslant c_0 + 1$ and $c_0 \geqslant a_0 + 1$, thus $b_0 \geqslant (a_0) + 1) + 1 = a_0 + 2$ and thus, $b_0 - a_0 \geqslant 2$, while we have $b_0 - a_0 = 1$. This contradiction shows that such an event c cannot exist and thus, that indeed

$$a \prec b.$$

4.2°. Let us now prove that if $a \prec b$, then $b_0 - a_0 = 1$. In this case, the inequality $|a_1 - b_1| \leqslant 1$ follows from the formula (2).

We will prove the desired result by contradiction. Indeed, let us assume that $b_0 - a_0 \geqslant 2$. In this case:

- in the formula (2), we can have either equality or strict inequality, and
- the difference $a_1 - b_1$, can be either non-negative or non-positive.

Let us consider all $2 \times 2 = 4$ combinations of these cases.

4.2.1°. Let us first consider the case when $b_0 - a_0 = |a_1 - b_1|$ and $a_1 - b_1 \geqslant 0$. In this case, $b_0 - a_0 = a_1 - b_1 \geqslant 0$. Then, for $c = (b_0 - 1, b_1 + 1)$, we clearly have $c \leqslant b$ and $c \neq b$, hence $c < b$.

We also have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 = (a_1 - b_1) - 1 = a_1 - (b_1 + 1) =$$

$$a_1 - c_1,$$

so $c_0 - a_0 \geqslant |a_1 - c_1|$ and $a \leqslant c$. Since $b_0 - a_0 \geqslant 2$, we have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 \geqslant 1$$

hence $a \neq c$ and $a < c$.

So, we have $a < c < b$, which contradicts to our assumption that $a \prec c$.

4.2.2°. Let us now consider the case when $b_0 - a_0 = |a_1 - b_1|$ and $a_1 - b_1 \leqslant 0$. In this case, $b_0 - a_0 = b_1 - a_1 \geqslant 0$. Then, for $c = (b_0 - 1, b_1 - 1)$, we clearly have $c \leqslant b$ and $c \neq b$, hence $c < b$.

We also have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 = (b_1 - a_1) - 1 = (b_1 - 1) - a_1 =$$

$$c_1 - a_1,$$

so $c_0 - a_0 \geqslant |a_1 - c_1|$ and $a \leqslant c$. Since $b_0 - a_0 \geqslant 2$, we have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 \geqslant 1$$

hence $a \neq c$ and $a < c$.

So, we have $a < c < b$, which contradicts to our assumption that $a \prec c$.

4.2.3°. Let us now consider the case when $b_0 - a_0 > |a_1 - b_1|$ and $a_1 - b_1 \geqslant 0$. In this case, $b_0 - a_0 > a_1 - b_1 \geqslant 0$. Since we only consider integer coordinates, this implies that $b_0 - a_0 \geqslant a_1 - b_1 + 1$.

Then, for $c = (b_0 - 1, b_1)$, we clearly have $c \leqslant b$ and $c \neq b$, hence $c < b$.

We also have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 \geqslant (a_1 - b_1 + 1) - 1 = a_1 - b_1 =$$

$$a_1 - c_1 \geqslant 0,$$

so $c_0 - a_0 \geqslant |a_1 - c_1|$ and $a \leqslant c$. Since $b_0 - a_0 \geqslant 2$, we have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 \geqslant 1$$

hence $a \neq c$ and $a < c$.

So, we have $a < c < b$, which contradicts to our assumption that $a \prec c$.

4.2.4°. Finally, let us now consider the case when $b_0 - a_0 > |a_1 - b_1|$ and $a_1 - b_1 \leqslant 0$. In this case, $b_0 - a_0 > b_1 - a_1 \geqslant 0$. Since we only consider integer coordinates, this implies that $b_0 - a_0 \geqslant b_1 - a_1 + 1$.

Then, for $c = (b_0 - 1, b_1)$, we clearly have $c \leqslant b$ and $c \neq b$, hence $c < b$.

We also have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 \geqslant (b_1 - a_1 + 1) - 1 = b_1 - a_1 =$$

$$c_1 - a_1 \geqslant 0,$$

so $c_0 - a_0 \geqslant |a_1 - c_1|$ and $a \leqslant c$. Since $b_0 - a_0 \geqslant 2$, we have

$$c_0 - a_0 = (b_0 - 1) - a_0 = (b_0 - a_0) - 1 \geqslant 1$$

hence $a \neq c$ and $a < c$.

So, we have $a < c < b$, which contradicts to our assumption that $a \prec c$.

4.2.5°. In all four cases, we have a contradiction, so indeed, if $a \prec b$, then $b_0 - a_0 = 1$. Part 4 is thus proven.

5°. Let us now define "border" relation as follows:

$$a \ll b \Leftrightarrow (a \prec b \,\&\, \exists c\, (b \prec c \,\&\, \forall a\, (a \prec d \prec c \rightarrow d = b))).$$

Let us prove that $a \ll b$ if and only if $b_0 - a_0 = |b_1 - a_1| = 1$, i.e., in other words, that $a \ll b$ if and only i $b_0 - a_0 = 1$ and $b_1 \neq a_1$.

To prove this, we need to prove:

- that if $a \prec b$ and $a_1 = b_1$, then $a \not\ll b$, and
- that if $a \prec c$ and $a_1 \neq b_1$ then $a \ll b$.

Let us prove these two statements one by one.

5.1°. Let us first consider the case when $a \prec b$ and $b_1 = a_1$. In this case, as we have shown in Part 4 of this proof, $b_0 - a_0 = 1$. Let us show that in this case, for all c for which $b \prec c$, there exists $d \neq b$ for which $a \prec d \prec c$ – which means that $a \not\ll b$.

Indeed, according to Part 4, the condition $b \prec c$ is satisfied only for three elements c: $c = (b_0 + 1, b_1 - 1)$, $c = (b_0 + 1, b_1)$, and $c = (b_0 + 1, b_1 + 1)$. Let us consider all three cases one by one.

5.1.1°. When $c = (b_0 + 1, b_1 - 1)$, then, as one can check, for $d = (b_0, b_1 - 1) \neq b$, we have $d = (a_0 + 1, a_1 - 1)$, thus $a \prec d \prec c$.

5.1.2°. When $c = (b_0 + 1, b_1)$, then, as one can check, for $d = (b_0, b_1 - 1) \neq b$, we have $d = (a_0 + 1, a_1 - 1)$, thus $a \prec d \prec c$.

5.1.3°. When $c = (b_0 + 1, b_1 + 1)$, then, as one can check, for $d = (b_0, b_1 + 1) \neq b$, we have $d = (a_0 + 1, a_1 + 1)$, thus $a \prec d \prec c$.

5.1.4°. In all three cases, we indeed have $a \not\ll b$.

5.2°. Let us now prove that if $a \prec b$ and $b_1 \neq a_1$, i.e., if $b_0 - a_0 = 1$ and $|b_1 - a_1| = 1$, then $a \ll c$. Let us consider two possible cases: $b_1 - a_1 = 1$ and $b_1 - a_1 = -1$.

5.2.1°. When $b_0 - a_0 = b_1 - a_1 = 1$, i.e., when $b = (a_0 + 1, a_1 + 1)$, we can take

$$c = (b_0 + 1, b_1 + 1) = (a_0 + 2, a_1 + 2).$$

In this case, if $a \prec d \prec c$, then $c_0 - a_0 = (c_0 - d_0) + (d_0 - a_0) = 2$. Since $a \prec d \prec c$, each of the two terms $c_0 - d_0$ and $d_0 - a_0$ cannot exceed 1, so the only way for their sum to be equal to 2 is when both are equal to 1, i.e., when $d_0 - a_0 = 1$ and thus,

$$d_0 = a_0 + 1 = b_0.$$

Similarly, we have $c_1 - a_1 = (c_1 - d_1) + (d_1 - a_1) = 2$. Since $a \prec d \prec c$, each of the two terms $c_1 - d_1$ and $d_1 - a_1$ cannot exceed 1, so the only way for their sum to be equal to 2 is when both are equal to 1, i.e., when $d_1 - a_1 = 1$ and thus,

$$d_1 = a_1 + 1 = b_1.$$

Here, $d_0 = b_0$ and $d_1 = b_1$, hence indeed $d = b$, thus $a \ll b$.

5.2.2°. When $b_0 - a_0 = a_1 - b_1 = 1$, i.e., when $b = (a_0 + 1, a_1 - 1)$, we can take

$$c = (b_0 + 1, b_1 - 1) = (a_0 + 2, a_1 - 2).$$

In this case, if $a \prec d \prec c$, then $c_0 - a_0 = (c_0 - d_0) + (d_0 - a_0) = 2$. Since $a \prec d \prec c$, each of the two terms $c_0 - d_0$ and $d_0 - a_0$ cannot exceed 1, so the only way for their sum to be equal to 2 is when both are equal to 1, i.e., when $d_0 - a_0 = 1$ and thus,

$$d_0 = a_0 + 1 = b_0.$$

Similarly, we have $a_1 - c_1 = (a_1 - d_1) + (d_1 - c_1) = 2$. Since $a \prec d \prec c$, each of the two terms $a_1 - d_1$ and $d_1 - c_1$ cannot exceed 1, so the only way for their sum to be equal to 2 is when both are equal to 1, i.e., when $a_1 - d_1 = 1$ and thus,

$$d_1 = a_1 - 1 = b_1.$$

Here, $d_0 = b_0$ and $d_1 = b_1$, hence indeed $d = b$. Thus indeed $a \ll b$.

Part 5 is proven.

6°. Since the relation $\prec$ is defined in terms of $\leqslant$ and the relation $\ll$ is defined in terms of $\prec$ and $\leqslant$, both relation $\prec$ and $\ll$ are preserved for each causality-preserving transformation.

According to Part 5, we have $(0,0) \ll (1,1)$, thus

$$(0,0) = f((0,0)) \ll f((1,1)).$$

Hence, either $f((1,1)) = (1,1)$ or $f((1,1)) = (1,-1)$. In the second case, we can apply a reflection and get $f((1,1)) = (1,1)$. So, without losing generality, we can assume that $f((0,0)) = (0,0)$ and $f((1,1)) = (1,1)$.

From the fact that $(0,0) \ll (1,-1)$, we conclude that $(0,0) \ll f((1,-1))$, thus $f((1,-1))$ is equal either to $(1,1)$ or to $(1,-1)$. Since f is a bijection and $f((1,1)) = (1,1)$, we cannot have $f((1,-1)) = (1,1)$, thus we have

$$f((1,-1)) = (1,-1).$$

7°. Let $a \ll b \ll c$. In this case, according to Part 5 of this proof, we have $b_1 - a_1 = \pm 1$ and $c_1 - b_1 = \pm 1$. We say that a, b, and c go *in the same direction* if $a \prec d \prec c$ implies that $d = b$. Let us show that they go in the same direction if and only if $b_1 - a_1 = c_1 - b_1$ – i.e., if and only if these differences have the same sign.

Indeed:

- if the signs are the same, then the proof is similar to Part 5.2;
- on the other hand, if the differences have different signs, then $c_1 = a_1$, so for $d = (a_0 + 1, a_1) \neq b$, we have $a \prec d \prec c$.

8°. Let us now prove that $f(x) = x$ for all $x \geqslant (0,0)$. Let us consider two possible cases: $x_1 \geqslant 0$ and $x_1 < 0$.

8.1°. If $x_1 \geqslant 0$, then we have

$$(0,0) \ll (1,1) \ll \ldots \ll (x_1, x_1) \prec (x_1 + 1, x_1) \prec \ldots \prec (x_0, x_1),$$

where the first x_1 relations go in the same direction, and the remaining ones are not border relations. All these relations are preserved under any causality-preserving transformation f. Thus, taking into account that $f((0,0)) = (0,0)$ and $f((1,1)) = (1,1)$, we have

$$(0,0) \ll (1,1) \ll f((2,2)) \ldots \ll f((x_1, x_1)) \prec$$

$$f((x_1 + 1, x_1)) \prec \ldots \prec f((x_0, x_1)),$$

where the first x_1 relations go in the same direction, and the remaining ones are not border relations.

Since $(1,1) \ll f((2,2))$, we conclude that $f((2,2))$ is either $(2,0)$ or $(2,2)$. Since $(0,0) \ll (1,1) \ll f((2,2))$ go in the same direction, the differences must have the same sign, so we must have $f((2,2)) = (2,2)$. Similarly, we prove that $f((3,3)) = (3,3)$, etc., and $f((x_1, x_1)) = (x_1, x_1)$. Now, in the remaining cases, we have immediately following relations which are not border relations, this means that a_0 increases by 1, and a_1 remains the same. Thus, we have

$$f((x_1 + 1, x_1)) = (x_1 + 1, x_1),$$

etc., all the way to the desired equality $f((x_0, x_1)) = (x_0, x_1)$.

8.2°. If $x_1 < 0$, then we similarly have

$$(0,0) \ll (1,-1) \ll \ldots \ll (|x_1|, x_1) \prec (|x_1|+1, x_1) \prec \ldots \prec (x_0, x_1),$$

where the first x_1 relations go in the same direction, and the remaining ones are not border relations. All these relations are preserved under any causality-preserving transformation f. Thus, taking into account that $f((0,0)) = (0,0)$ and $f((1,-1)) = (1,-1)$, we have

$$(0,0) \ll (1,-1) \ll f((2,-2)) \ldots \ll f((|x_1|, x_1)) \prec$$

$$f((|x_1|+1, x_1)) \prec \ldots \prec f((x_0, x_1)),$$

where the first $|x_1|$ relations go in the same direction, and the remaining ones are not border relations.

Since $(1,-1) \ll f((2,-2))$, we conclude that $f((2,-2))$ is either $(2,0)$ or $(2,-2)$. Since $(0,0) \ll (1,-1) \ll f((2,2))$ go in the same direction, the differences must have the same sign, so we must have $f((2,-2)) = (2,-2)$. Similarly, we prove that $f((3,-3)) = (3,-3)$, etc., and $f((|x_1|, x_1)) = (|x_1|, x_1)$. Now, in the remaining cases, we have immediately following relations which are not border relations, this means that a_0 increases by 1, and a_1 remains the same. Thus, we have

$$f((|x_1|+1, x_1)) = (|x_1|+1, x_1),$$

etc., all the way to the desired equality $f((x_0, x_1)) = (x_0, x_1)$.

9°. We proved that $f(x) = x$ for all $x \geqslant 0$. Now, we can similarly prove that for any $y \leqslant x$, we have $f(y) = y$, i.e., that indeed $f(y) = y$ for all y. The original transformation is a composition of this transformation f, shift, and possibly reflection, so the proposition is proven.

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References

1. Alexandrov A.D. On Lorentz transformations // Uspekhi Math. Nauk. 1950, vol. 5, no. 1, P. 187 (in Russian).
2. Alexandrov A.D., Ovchinnikova V.V. Remarks on the foundations of Special Relativity // Leningrad University Vestnik. 1953, no. 11. P. 94–110 (in Russian).

3. Zeeman E.C. Causality implies the Lorentz group // Journal of Mathematical Physics. 1964, vol. 5, no. 4. P. 490–493.

ДИСКРЕТНАЯ ПРИЧИННОСТЬ ВЛЕЧЕТ ГРУППУ ЛОРЕНЦА

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Аннотация. Известно, что для пространства-времени Минковского размерности больше 2 любое сохраняющее причинность преобразование линейно. Также известно, что в двумерном пространстве-времени существует множество нелинейных преобразований, сохраняющих причинность. В этой статье мы показываем, что для двумерного пространства-времени, если мы ограничиваемся дискретным пространством-временем, линейность сохраняется: только линейное преобразование сохраняет причинность.

Ключевые слова: причинность, специальная теория относительности, теорема Александрова-Зеемана, дискретное пространство-время.

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