

PHYSICAL TRAJECTORIES ARE SMOOTH, WITH VELOCITIES AT LEAST AS CONTINUOUS AS BROWNIAN MOTION

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Abstract. The fact that the kinetic energy of a particle cannot exceed its overall energy implies that the velocity – i.e. the derivative of the trajectory – should be bounded. This means, in effect, that all the trajectories are differentiable (smooth). However, at first glance, there seems to be no direct requirement that the velocities continuously depend on time. In this paper, we show that the properties of electromagnetic field necessitate that the velocities are continuous functions of time – moreover, that they are at least as continuous as the Brownian motion.

Keywords: smooth trajectories, continuous dependence, Brownian motion.

1. Formulation of the Problem

It is important to know which physical quantities are smooth and which are continuous. From the physical viewpoint, there is a big difference between smooth changes and abrupt changes – as, e.g., in phase transitions. From the mathematical viewpoint, an abrupt change means that either some physical quantity is discontinuous or that its derivative is discontinuous. It is therefore important to analyze which discontinuities are physically possible.

We can view this question from a more mathematical viewpoint. Namely, physical processes are usually described by differential equations. It is known that the properties of the solutions to differential equations often depend on whether the corresponding sources and/or initial conditions are continuous or differentiable. So, to understand physical phenomena, it is important to know which physical quantities are smooth and which are continuous.

Fundamental case: particles. From the physical viewpoint, everything consists of particles (see, e.g., [1, 2]):

- a seemingly continuous solid body consists of atoms,
- a seemingly continuous field consists of field quanta – e.g., electromagnetic field consists of photons, etc.

Each quantity of a physical system is a combination of quantities of the constituent particles. Thus, the general question of what is smooth and what is continuous in the physical world can be reduced to the fundamental case of checking when is a trajectory of each particle continuous or smooth.

Trajectories should be smooth. A kinetic energy of a particle

$$m \cdot \frac{v^2}{2}$$

of mass m cannot exceed its overall energy. Thus, the velocities v are bounded. This means, in effect, that all the trajectories are differentiable (smooth).

But are velocities continuous? At first glance, it seems that velocities – i.e., derivatives of the trajectories – can be discontinuous: e.g., in a collision of two point particle, they abruptly change their velocities.

What we do in this paper. In this paper, we show that the above at-first-glance analysis is misleading.

Namely, if we take into account simple properties of such an ubiquitous thing as the electromagnetic field, then we can conclude that velocities have to be continuous – at least as continuous as Brownian motion.

2. Analysis of the Problem and the Resulting Conclusion

Basic physics: reminder. Matter largely consists of electrons and hadrons – that, in turn, consists of quarks. Both electrons and quarks are electrically charged, and thus, most follow the equations of electrodynamics.

According to these equations, every charged particle accelerating with acceleration a radiates energy – and its energy loss proportional to a^2 ; see, e.g., [1, 2]. Thus, the overall radiation-based loss of energy between two moments $t_1 < t_2$ is proportional to the integral

$$\int_{t_1}^{t_2} (a(t))^2 dt.$$

Analyzing the problem. Overall, a particle cannot lose more energy than it originally had + what it got, so for each particle, we must have

$$\int_{t_1}^{t_2} (a(t))^2 dt \leq E_0$$

for some constant E_0 .

It is known that the scalar (dot) product

$$\sum_{i=1}^n x_i \cdot y_i$$

of two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ cannot exceed the product of their lengths:

$$\sum_{i=1}^n x_i \cdot y_i \leq \sqrt{\sum_{i=1}^n x_i^2} \cdot \sqrt{\sum_{i=1}^n y_i^2}.$$

In the limit when sum tends to an integral, this implies that for every two functions $x(t)$ and $y(t)$, we have

$$\int_{t_1}^{t_2} x(t) \cdot y(t) dt \leq \sqrt{\int_{t_1}^{t_2} (x(t))^2 dt} \cdot \sqrt{\int_{t_1}^{t_2} (y(t))^2 dt}.$$

In particular, for $x(t) = |a(t)|$, $y(t) = 1$, and $t_2 = t_1 + \Delta t$, the second integral is equal to Δt , so we have

$$\int_{t_1}^{t_1+\Delta t} |a(t)| dt \leq \sqrt{\int_{t_1}^{t_1+\Delta t} (a(t))^2 dt} \cdot \sqrt{\Delta t}.$$

The first integral in the right-hand side is bounded by E_0 , so

$$\int_{t_1}^{t_1+\Delta t} |a(t)| dt \leq \sqrt{E_0} \cdot \sqrt{\Delta t}.$$

Here, we have

$$|v(t_1 + \Delta t) - v(t_1)| = \left| \int_{t_1}^{t_1+\Delta t} a(t) dt \right| \leq \int_{t_1}^{t_1+\Delta t} |a(t)| dt,$$

thus

$$|v(t_1 + \Delta t) - v(t_1)| \leq c \cdot \sqrt{\Delta t}, \quad (1)$$

where we denoted $c \stackrel{\text{def}}{=} \sqrt{E_0}$.

Velocities are continuous. According to the formula (1), when $\Delta t \rightarrow 0$, we have $|v(t_1 + \Delta t) - v(t_1)| \rightarrow 0$, i.e., $v(t_1 + \Delta t) \rightarrow v(t_1)$.

Thus, each velocity is indeed a continuous function of time.

Velocities are at least as continuous as the Brownian motion. For the Brownian motion, the average value of the difference $|v(t_1 + \Delta t) - v(t_1)|$ is indeed proportional to $\sqrt{\Delta t}$; see, e.g., [1, 2].

In this sense, the velocities are indeed as continuous as the Brownian motion.

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REFERENCES

1. Feynman R., Leighton R., and Sands M. The Feynman Lectures on Physics. Addison Wesley, Boston, Massachusetts, 2005.
2. Thorne K.S. and Blandford R.D. Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics. Princeton University Press, Princeton, New Jersey, 2017.

ФИЗИЧЕСКИЕ ТРАЕКТОРИИ ГЛАДКИЕ, СО СКОРОСТЯМИ, ПО КРАЙНЕЙ МЕРЕ, ТАКИМИ ЖЕ НЕПРЕРЫВНЫМИ КАК БРОУНОВСКОЕ ДВИЖЕНИЕ

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Аннотация. Тот факт, что кинетическая энергия частицы не может превышать её полную энергию, означает, что скорость, т. е. производная от траектории, должна быть ограничена. Фактически это означает, что все траектории дифференцируемы (гладки). Однако, на первый взгляд, кажется, что нет прямого требования, чтобы скорости непрерывно зависели от времени. В этой статье мы показываем, что свойства электромагнитного поля требуют, чтобы скорости были непрерывными функциями времени, более того, чтобы они были, по крайней мере, такими же непрерывными, как и броуновское движение.

Ключевые слова: гладкие траектории, непрерывная зависимость, броуновское движение.

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